Fluid Dynamics

- The study of fluid mechanics in which the forces and energies are considered along with the motion.

- Types of forces –
  - ✓ Body forces – proportional to volume or weight
    e.g. centrifugal, magnetic, gravity etc.
  - ✓ Surface forces – proportional to area e.g. pressure, shear or tangential forces
  - ✓ Line forces – proportional to length e.g. surface tension
- Similar to solid mechanics fluid dynamics is also governed by Newton's second law of motion.

Newton's Second law and Fluid

- Resultant force on any fluid element must be equal to the product of mass and the acceleration the element in the same direction

Total force

$$\sum F = Ma$$

$$\sum F_x = Ma_x$$

$$\sum F_y = Ma_y$$

$$\sum F_z = Ma_z$$

Force per unit volume

$$\sum f = \rho a$$

$$\sum f_x = \rho a_x$$

$$\sum f_y = \rho a_y$$

$$\sum f_z = \rho a_z$$

a is the acceleration of fluid particle, F is the total force acting on fluid particle and ρ is the mass density; x, y and z indicate the suffix for the components along respective directions

Various forces acting on the fluid

- Gravity Force $F_g = Mg$
- Pressure Force, F_p
- Viscous force, F_v
- Turbulent Force, F_t
- Surface Tension, F_s
- Compressibility Force, F_e

$$Ma = F_g + F_p + F_v + F_t + F_s + F_e$$

$$Ma_x = F_{gx} + F_p x + F_{vx} + F_{tx} + F_{sx} + F_{ex}$$

$$Ma_y = F_{gxy} + F_{py} + F_{vy} + F_{ty} + F_{sy} + F_{ey}$$

$$Ma_z = F_{gz} + F_{pz} + F_{vz} + F_{tz} + F_{sz} + F_{ez}$$

Reynolds Equations of fluid motion

For fluid in motion the forces due to surface tension and the compressibility effects are negligible

$$Ma = F_g + F_p + F_v + F_t$$

$$Ma_x = F_{gx} + F_p x + F_{vx} + F_{tx}$$

$$Ma_y = F_{gxy} + F_{py} + F_{vy} + F_{ty}$$

$$Ma_z = F_{gz} + F_{pz} + F_{vz} + F_{tz}$$

These equations are useful for the analysis of turbulent flow

Navier-Stokes Equations

For laminar or viscous flow the forces due to turbulence are negligible

$$Ma = F_g + F_p + F_v$$

$$Ma_x = F_{gx} + F_p x + F_{vx}$$

$$Ma_y = F_{gxy} + F_{py} + F_{vy}$$

$$Ma_z = F_{gz} + F_{pz} + F_{vz}$$

These equations are useful for the analysis of laminar flow

Euler's Equations for fluid flow

For ideal fluid flow the viscous forces are negligible

$$Ma = F_g + F_p$$

$$Ma_x = F_{gx} + F_p x$$

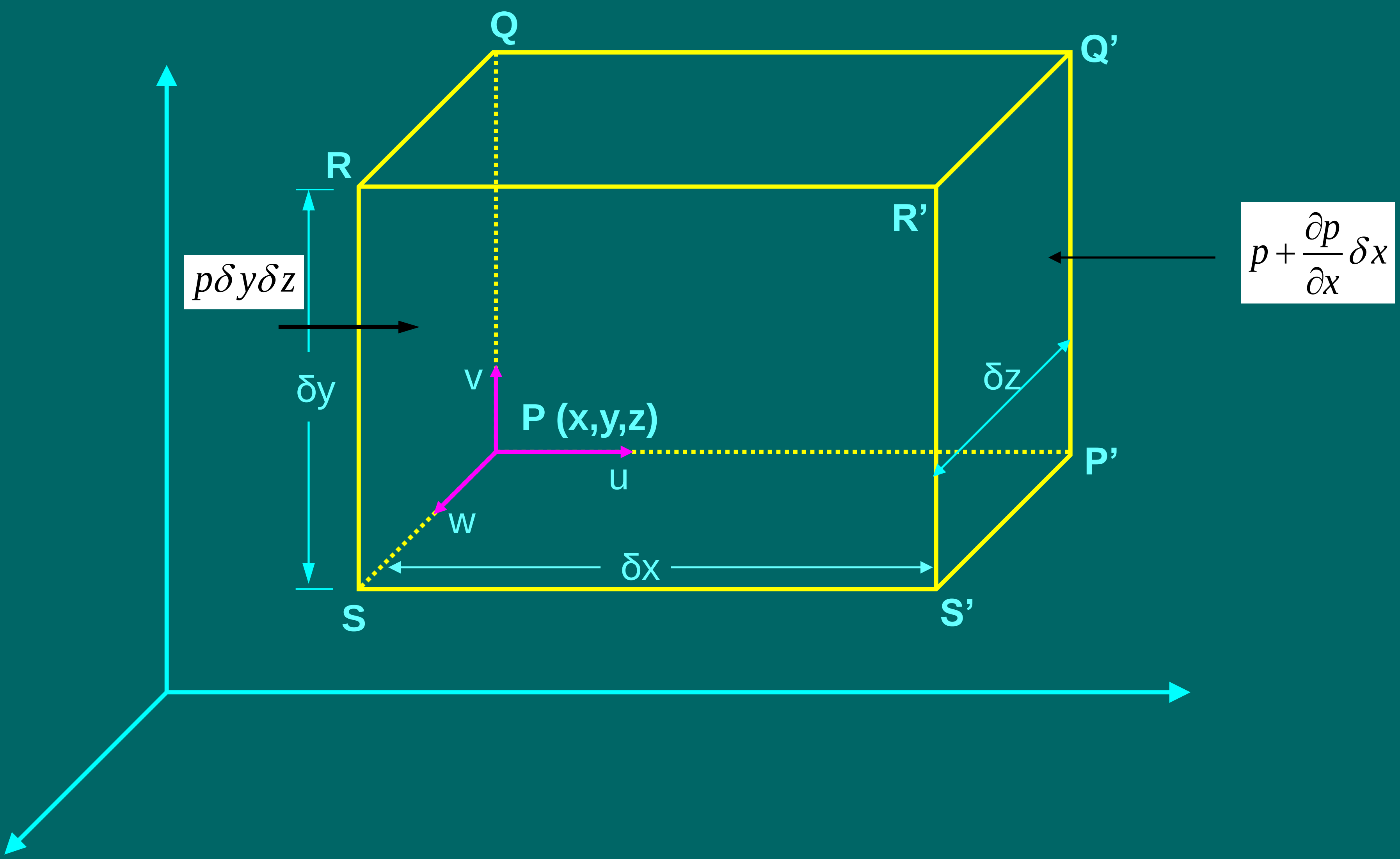
$$Ma_y = F_{gxy} + F_{py}$$

$$Ma_z = F_{gz} + F_{pz}$$

The Euler's equation is comprised of only body and surface forces

These equations are useful if the viscosity of the fluid is negligible or insignificant.

Derivation of Euler's Equation



Derivation of Euler's Equation

- Let X, Y and Z be the components of body forces per unit mass at point P
- Mass of parallelepiped = (ρδxδyδz)
- Total component of body force in X-dir^n = X (ρδxδyδz)
- Total component of body force in Y-dir^n = Y (ρδxδyδz)
- Total component of body force in Z-dir^n = Z (ρδxδyδz)
- Pressure intensity at P = p
- Total pressure force acting on PQRS = p (δyδz)
- Pressure intensity on P'Q'R'S' = $p + \frac{\partial p}{\partial x}\delta x$
- Total pressure force acting on PQRS = $(p + \frac{\partial p}{\partial p}\delta x)\delta y\delta z$

Derivation of Euler's Equation

- **Net pressure force in X-dir^n**

$$F_{px} = p\delta y\delta z - \left(p + \frac{\partial p}{\partial x}\delta x\right)\delta y\delta z \Rightarrow F_{px} = -\frac{\partial p}{\partial x}\delta x\delta y\delta z$$

Similarly,

$$F_{py} = -\frac{\partial p}{\partial y}\delta x\delta y\delta z \quad \textbf{and} \quad F_{pz} = -\frac{\partial p}{\partial z}\delta x\delta y\delta z$$

Adding body forces and pressure forces in X-direction and equating to the product of mass and acceleration in X-direction

$$X(\rho\delta x\delta y\delta z) - \frac{\partial p}{\partial x}\delta x\delta y\delta z = \rho(\delta x\delta y\delta z)a_x \quad \text{---- A}$$

Derivation of Euler's Equation

- Thus

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = a_x$$

- Similarly,

$$Y - \frac{1}{\rho}\frac{\partial p}{\partial y} = a_y$$

$$Z - \frac{1}{\rho}\frac{\partial p}{\partial z} = a_z$$

Euler's equations of fluid motion

Where,

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Derivation of Euler's Equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+X$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+Y$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+Z$$

No assumption that ρ is constant is made in these equations, therefore these are applicable to compressible or incompressible, non-viscous fluid flow in steady as well as unsteady state

Integration of Euler's Equation

- **Gives energy equation under following assumptions**

1. **There exists a force potential which is defined as the function whose negative derivative wrt any direction gives the component of body force per unit mass in that direction (Ω). Thus,**

$$X = -\frac{\partial \Omega}{\partial x} \quad Y = -\frac{\partial \Omega}{\partial y} \quad Z = -\frac{\partial \Omega}{\partial z}$$

2. **The flow is irrotational i.e. velocity potential exists or the flow may be rotational but it's steady**

Integration of Euler's Equation

When flow is irrotational

- **Consider Euler's equation in X-direction**

$$X - \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

- **If flow is irrotational** ⟹ $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}; \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$

- **Existence of velocity potential** ⟹ $u = -\frac{\partial \phi}{\partial x}$

Therefore, $$-\frac{\partial \Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial^2 \phi}{\partial x \partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x}$$

Integration of Euler's Equation

Thus,
$$\frac{\partial}{\partial x}\left[\Omega+\frac{p}{\rho}+\frac{u^2+v^2+w^2}{2}-\frac{\partial\phi}{\partial t}\right]=0$$

On integrating wrt x
$$\left[\Omega+\frac{p}{\rho}+\frac{V^2}{2}-\frac{\partial\phi}{\partial t}\right]=F_1(y,z,t)$$

Similarly, for Y and Z directions
$$\left[\Omega+\frac{p}{\rho}+\frac{V^2}{2}-\frac{\partial\phi}{\partial t}\right]=F_2(x,z,t)$$

$$\left[\Omega+\frac{p}{\rho}+\frac{V^2}{2}-\frac{\partial\phi}{\partial t}\right]=F_3(x,y,t)$$

Thus,
$$F_1(y,z,t)=F_2(x,z,t)=F_3(x,y,t)$$

Since x, y and z are the independent variables the above equation will hold good only if these variables disappear from functional term and F is only function of t

Integration of Euler's Equation

Therefore, $$\left[\Omega + \frac{p}{\rho} + \frac{V^2}{2} - \frac{\partial \phi}{\partial t}\right] = F(t)$$

For steady flow, $$\left[\Omega + \frac{p}{\rho} + \frac{V^2}{2}\right] = C$$

If the body force exerted on the fluid is only due to gravity and if Z axis is so oriented that z is measured in vertical direction with reference to datum then,

$$-\frac{\partial \Omega}{\partial x} = 0; -\frac{\partial \Omega}{\partial y} = 0; -\frac{\partial \Omega}{\partial z} = -g$$

Since g is the force per unit mass and can be +ve when acting in downward direction $$\Omega = gz + C_1$$ **At z = 0 Ω = 0, therefore C_1 = 0**

Therefore, $$\Omega = gz$$

Integration of Euler's Equation

$$\left[\frac{p}{\rho}+\frac{V^2}{2}+gz\right]=C$$

Bernoulli's Equation

$$\left[\frac{p}{\rho g}+\frac{V^2}{2g}+z\right]=C$$

- **Energy Equation as each term indicates energy per unit weight**

$$\left[\frac{p}{w}+\frac{V^2}{2g}+z\right]=C$$

- **Each term also indicate the head e.g. pressure, velocity and potential head**

Thus, the Bernoulli's theorem states that for steady flow of an incompressible fluid the total energy which is the summation of pressure, velocity and potential energy remains constant at any point

Integration of Euler's Equation

$\dfrac{p}{w}$ ⇨ **Pressure head;** $\dfrac{V^2}{2g}$ ⇨ **Velocity head or kinetic head;**

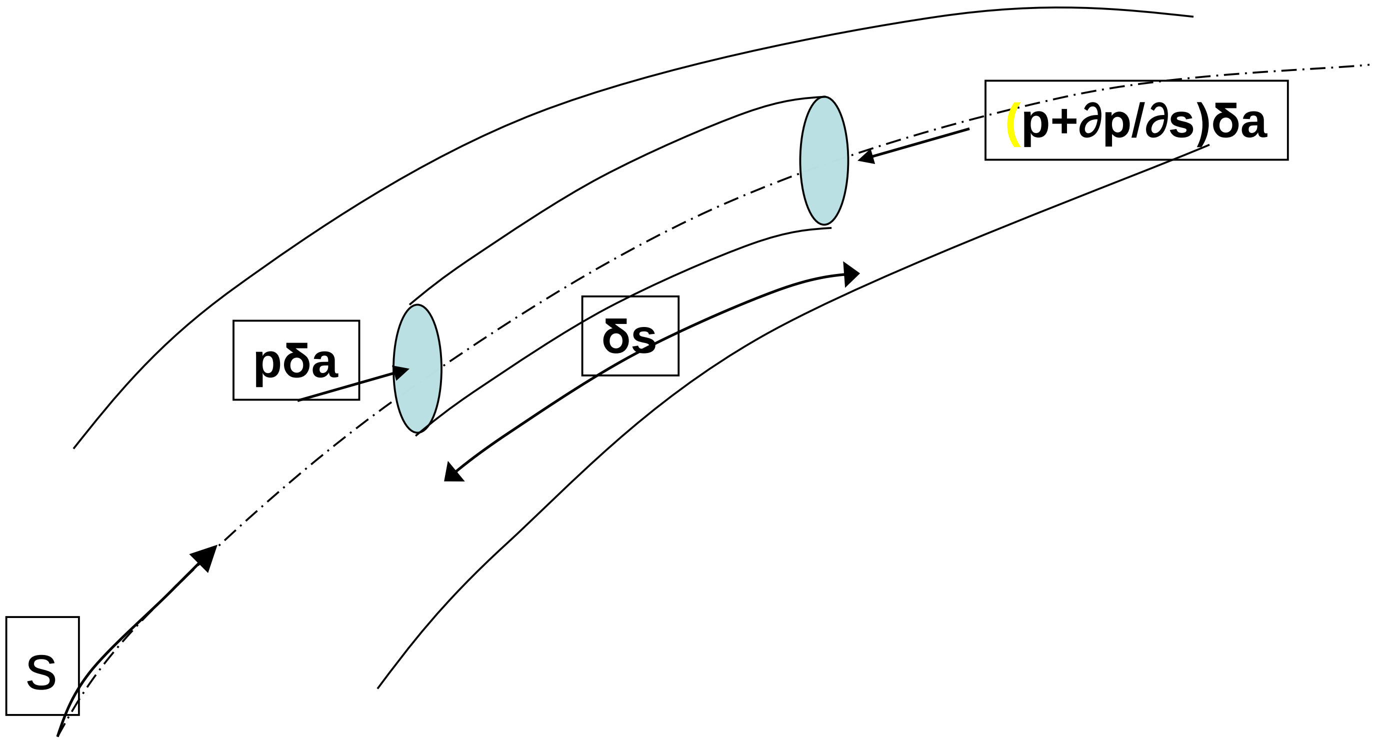
z ⇨ **Potential head or datum head**

$\left(\dfrac{p}{w}+z\right)$ ⇨ **Piezometric head**

$$\left[\frac{p_1}{w}+\frac{V_1^{\,2}}{2}+z_1\right]=\left[\frac{p_2}{w}+\frac{V_2^{\,2}}{2}+z_2+h_L\right]$$

Integration of Euler's Equation

- When flow is rotational but steady
- Consider an element of stream filament of CSA δa



Integration of Euler's Equation

- Let S be the body force per unit mass along streamline
- Mass of fluid element = (ρδa δs)
- Total body force along S-direction = S (ρδa δs)
- Total pressure force on left end = pδa
- Total pressure force on right end = [p+(**∂p/∂s)** δs**]** δa

$$F_{ps} = [(p\delta a) - (p + \frac{\partial p}{\partial s}\delta s)\delta a] = -\frac{\partial p}{\partial s}\delta s \delta a$$

Steady flow acceleration $$a_s = V\frac{\partial V}{\partial s} = \frac{1}{2}\frac{\partial V^2}{\partial s}$$

Newton's second law of motion ⇨

$$S(\rho\delta s\delta a) - \frac{\partial p}{\partial s}\delta s\delta a = (\rho\delta s\delta a)\frac{1}{2}\frac{\partial V^2}{\partial s}$$

Integration of Euler's Equation

$$S = -\frac{\partial \Omega}{\partial s}$$

Therefore,

$$\frac{\partial \Omega}{\partial s} + \frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{1}{2}\frac{\partial V^2}{\partial s} = 0$$

$$\Omega + \frac{p}{\rho} + \frac{V^2}{2} = C$$

For incompressible flow

For compressible flow

$$\Omega + \int \frac{\partial p}{\rho} + \frac{V^2}{2} = C$$

$$\Omega = gz$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C$$

or

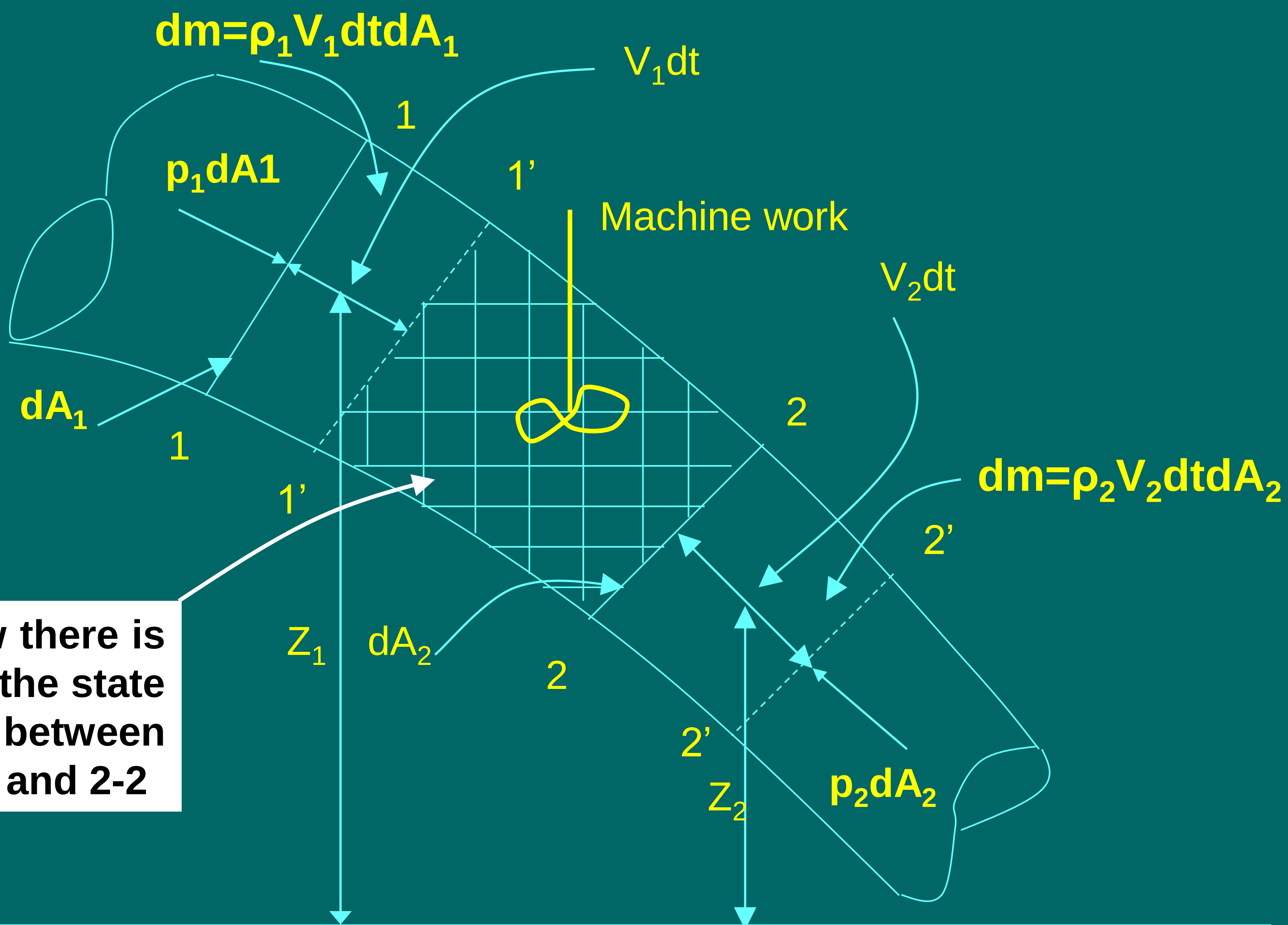
$$\frac{p}{w} + \frac{V^2}{2} + z = C$$

Euler's Equation – an important point

- If the flow is irrotational then the same Bernoulli’s equation is applicable to all the points in the flow field i. e. for all the streamlines the value of constant is same.

- For rotational flow the Bernoulli's equation is applicable for a particular streamline i. e. the value of constant is different for different streamlines

Bernoulli's Equation- Principle of conservation of energy



In steady flow there is no change in the state of fluid between Sections 1'-1' and 2-2

Figure:- Free body of flowing fluid occupying a portion of stream tube between two arbitrarily chosen sections 1-1 and 2-2

Derivation of BE using conservation of energy

- The general energy equation

$$\begin{bmatrix}\text{work done on the fluid}\\ \text{by force during dt}\end{bmatrix} \pm \begin{bmatrix}\text{Mechanical work}\\ \text{performed on the fluid}\end{bmatrix} = \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ 1'-1' \text{ and } 2'-2' \text{ at } t_2\end{bmatrix} - \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ \text{1-1 and 2-2 at } t_2\end{bmatrix}$$

- **But**

$$\begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ 1'-1' \text{ and } 2'-2' \text{ at } t_2\end{bmatrix} = \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ \text{1'-1' and 2-2 at } t_2\end{bmatrix} + \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ \text{2-2 and } 2'\text{-}2' \text{ at } t_2\end{bmatrix}$$

$$\text{and} \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ 1-1 \text{ and } 2-2 \text{ at } t_1\end{bmatrix} = \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ \text{1-1 and } 1'\text{-}1' \text{ at } t_1\end{bmatrix} + \begin{bmatrix}\text{Total energy of fluid}\\ \text{between}\\ 1'\text{-}1' \text{ and 2-2 at } t_1\end{bmatrix}$$

Derivation of BE using conservation of energy

For steady flow the state of the flowing fluid in the stream tube within the region bounded by sections 1'-1' and 2-2 remains unchanged wrt to time. Thus,

$$\begin{bmatrix}\textbf{Total energy of fluid}\\ \textbf{between}\\ \mathbf{1'-1'\ and\ 2-2\ at\ t_2}\end{bmatrix} = \begin{bmatrix}\textbf{Total energy of fluid}\\ \textbf{between}\\ \mathbf{1'\text{-}1'\ and\ 2\text{-}2\ at\ t_1}\end{bmatrix}$$

Hence the general energy equation for steady flow of fluid is reduced to

$$\begin{bmatrix}\textbf{work done on the fluid}\\ \textbf{by external forces during}\\ \textbf{dt}\end{bmatrix} \pm \begin{bmatrix}\textbf{Mechanical work}\\ \textbf{performed on the}\\ \textbf{fluid}\end{bmatrix}$$

$$= \begin{bmatrix}\textbf{Total energy of}\\ \textbf{fluid between}\\ \mathbf{2-2\ and\ 2'-2'\ at\ t_2}\end{bmatrix} - \begin{bmatrix}\textbf{Total energy of}\\ \textbf{fluid between}\\ \mathbf{1\text{-}1\ and\ 1'\text{-}1'\ at\ t_1}\end{bmatrix}$$

Derivation of BE using conservation of energy

- For steady flow the mass flow at sections 1-1 and 2-2 during dt being same

$$\begin{bmatrix} \textbf{Fluid mass between} \\ \mathbf{1}-1 \textbf{ and } \mathbf{1}^{'}-1^{'} \end{bmatrix} = \begin{bmatrix} \textbf{Fluid mass between} \\ \mathbf{2}-2 \textbf{ and } \mathbf{2}^{'}-2^{'} \end{bmatrix}$$

Work done by pressure force (p_1dA_1) on the free body of the fluid in the stream tube during *dt is* + $(p_1dA_1)(V_1dt)$

Work done by pressure force (p_2dA_2) on the free body of the fluid in the stream tube during *dt is* - $(p_2dA_2)(V_2dt)$

Net work done performed by pressure forces on the free body of the fluid in the stream tube during *dt is* $[(p_1dA_1)(V_1dt) - (p_2dA_2)(V_2dt)]$

Derivation of BE using conservation of energy

If dm is the total mass of fluid flowing across section 1-1 during dt or if dm is actually the mass of fluid within 1-1 and 1'-1' or between 2-2 and 2'-2', then

$$(V_1 dA_1)dt = \frac{dm}{\rho_1}; \ \textbf{and} \ (V_2 dA_2)dt = \frac{dm}{\rho_2}$$

Thus net work done by pressure forces on the free body of fluid is

$$\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)dm$$

Total work done by the fluid or on the fluid by external device is $\pm h_m g dm$

Derivation of BE using conservation of energy

- Gravitational potential energies are given as $z_1 gdm$ and $z_2 gdm$ and net potential energy is $(z_1 - z_2)gdm$
- Kinetic energies are $(V_1^2/2)dm$; $(V_2^2/2)dm$ and net KE : $dm(V_1^2 - V_2^2)/2)$

•Thus, general energy equation is given as

$$\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)dm \pm h_m gdm = (z_2 - z_1)gdm + \frac{dm}{2}(V_2^2 - V_1^2)$$

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2 \pm h_L$$

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2 \pm h_L$$

Derivation of BE using conservation of energy

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \textbf{constant}$$

This equation is applicable to the steady flow of an incompressible fluid

Kinetic energy correction factor

- Velocity is assumed to be uniform over the entire cross-section in the derivation of BE
- KE at any section can be obtained by integration of KE of all the particles over the cross section
- If v is the local velocity through dA (a small area along cross section), the mass flow = ρvdA
- KE of fluid passing through dA = $(\rho v dA)v^2/2$
- Total KE possessed by the flowing fluid across entire cross section A is

$$\int_A \rho \frac{v^3}{2} dA = \frac{w}{2g} \int_A v^3 dA$$

- Convenient way to express KE is with the help of mean velocity of flow (V).

Kinetic energy correction factor

- However, the actual KE is greater than the KE computed using mean velocity (V)
- Hence the factor α is introduced such that KE is

$$\textbf{Actual KE} = \alpha \frac{w}{2g} A V^3$$

- Thus,

$$\alpha \frac{w}{2g} A V^3 = \frac{w}{2g} \int_A v^3 dA \quad \Rightarrow \quad \alpha = \frac{1}{AV^3} \int_A v^3 dA$$

- Mathematically, the cube of average is less than the average of cubes i.e. $V^3 < \frac{1}{A} \int_A v^3 dA$ ⇒ α is always greater than 1

Kinetic energy correction factor

- For turbulent flow α lies between 1.03 to 1.06 which is close to 1.
- For laminar flow in pipes it is 2.

- The KE in BE can be introduced as

$$\frac{p_1}{w} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

Assignment – Prove that for laminar flow α = 2

BE for COMPRESSIBLE FLUID

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = C$$

Isothermal Change

$$\frac{p}{\rho} = K \textbf{ or } \frac{p}{\rho g} = K'$$

$$\rho = p / K$$

Substitute the value of ρ and integrate the pressure term

$$K \log_e p + \frac{V^2}{2} + gz = C$$

or

$$gK' \log_e p + \frac{V^2}{2} + gz = C$$

$$K \log_e p_1 + \frac{V_1^2}{2} + gz_1 = K \log_e p_2 + \frac{V_2^2}{2} + gz_2$$

$$gK' \log_e p_1 + \frac{V_1^2}{2} + gz_1 = gK' \log_e p_2 + \frac{V_2^2}{2} + gz_2$$

BE for COMPRESSIBLE FLUID

$$K \log_e (p_1 / p_2) = \frac{V_2^2}{2} - \frac{V_1^2}{2} + g(z_2 - z_1)$$

$$K^{'} \log_e (p_1 / p_2) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} + (z_2 - z_1)$$

BE for COMPRESSIBLE FLUID

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = C$$

Adiabatic change

$$\frac{p}{\rho^k} = C_1; \ \textbf{or} \ \frac{p}{(\rho g)^k} = C_1'$$

k is the exponent of adiabatic change

$$\frac{dp}{d\rho} = C_1 k \rho^{k-1}; \ \textbf{or} \ \frac{dp}{\rho} = C_1 k \rho^{k-2} d\rho$$

$$\frac{dp}{d\rho} = C_1' g^k k \rho^{k-1}; \ \textbf{or} \ \frac{dp}{\rho} = C_1' g^k k \rho^{k-2} d\rho$$

$$C_1 k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

$$C_1' g^k k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

BE for COMPRESSIBLE FLUID

Adiabatic change

$$C_1 k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

$$\frac{k}{k-1}\frac{p}{\rho} + \frac{V^2}{2} + gz = C$$

$$C_1' g^k k \frac{\rho^{k-1}}{(k-1)} + \frac{V^2}{2} + gz = C$$

$$\frac{k}{k-1}\frac{p}{w} + \frac{V^2}{2g} + z = C'$$

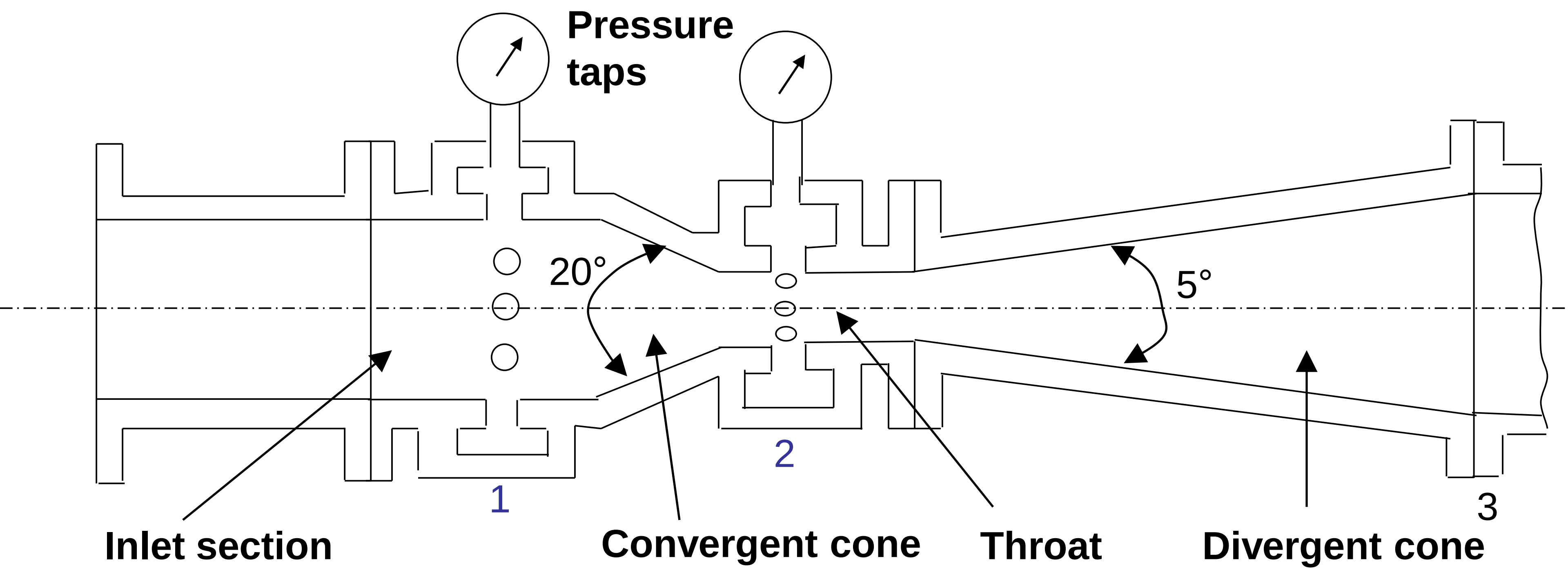
Applications of Bernoulli's Equation

- Discharge measurement
  - Venturi meter
  - Orifice meter
  - Nozzle meter
  - Rota meter
  - Elbow meter
- velocity measurement
  - Pitot tube

The other important equation used in these applications is continuity equation

Venturi Meter

- ➢ a device for measuring a flow rate through a pipe
- ➢G B Venturi – an Italian Physicist (1797)
- ➢Principle – reduction in cross-sectional area of flow passage creates pressure difference which enables flow measurement



Venturi Meter

- Convergent cone – angle -21°±1; length 2.7(D-d)
- Throat – length is d, diameter of throat is 1/3 to ¾ of dia of pipe
  - Diameter of throat is restricted by a cavitation
- Divergent cone – angle – 5° to 15° (preferably 6°)
  - length is much larger than convergent cone to avoid separation
- Let a_1 and a_2 be the cross sectional area of inlet section and sections at 2, respectively; p_1 and p_2 be corresponding pressures; V_1 and V_2 are respective velocities.
- Bernoulli's equation between 1 and 2 gives

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

Venturi Meter

- If Venturi is horizontal $z_1 = z_2$

$$\frac{p_1}{w}+\frac{V_1^2}{2g}=\frac{p_2}{w}+\frac{V_2^2}{2g} \Rightarrow \frac{p_1}{w}-\frac{p_2}{w}=\frac{V_2^2}{2g}-\frac{V_1^2}{2g}$$

$$\frac{p_1}{w}-\frac{p_2}{w}$$

Represents difference in pressure heads between inlet and throat which is known as Venturi head and denoted by h

Therefore,

$$h=\frac{V_2^2}{2g}-\frac{V_1^2}{2g}$$

Continuity equation,

$$Q_{th}=a_1V_1=a_2V_2 \Rightarrow V_1=\frac{Q_{th}}{a_1} \text{ and } V_2=\frac{Q_{th}}{a_2}$$

Hence

$$h=\frac{Q_{th}}{2g}\left[\frac{1}{a_2^2}-\frac{1}{a_1^2}\right] \Rightarrow Q_{th}=\frac{a_1a_2\sqrt{2gh}}{\sqrt{a_1^2-a_2^2}}$$

Coefficient of Discharge for Venturi Meter

- Actual discharge is always less than the theoretical discharge and hence actual discharge can be estimated by introducing a factor known as coefficient of discharge (C_d) such that

$$C_d \ \textbf{or} \ K = \frac{Q}{Q_{th}}$$

- Thus,

$$\mathbf{Q} = C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

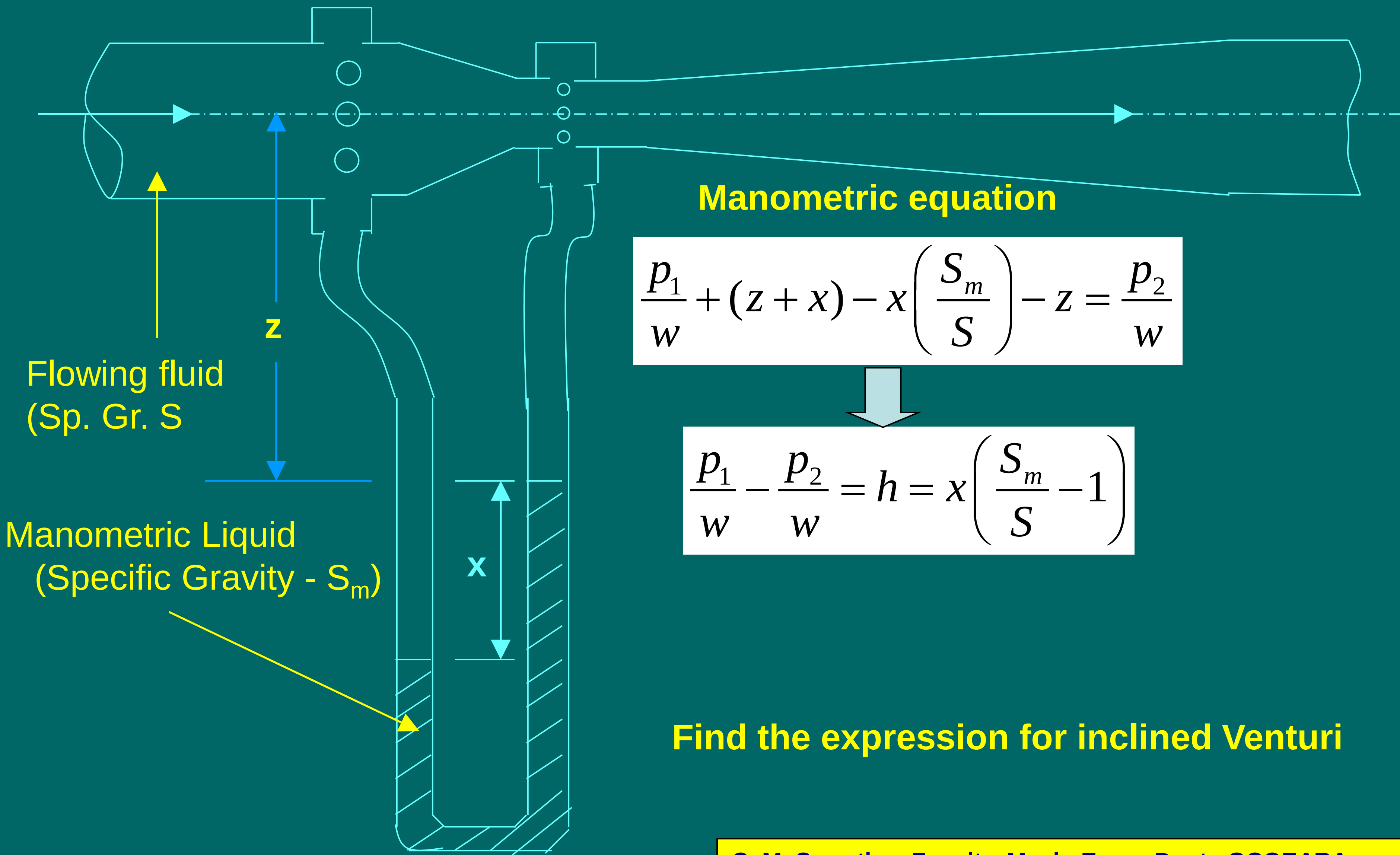
Constant of Venturi meter

$$C = \frac{a_1 a_2 \sqrt{2g}}{\sqrt{a_1^2 - a_2^2}}$$

$$Q = C_d C \sqrt{h}$$

Value of Cd for Venturi ranges between 0.95 to 0.98

Venturi Meter with U-tube manometer



Manometric equation

$$\frac{p_1}{w} + (z + x) - x\left(\frac{S_m}{S}\right) - z = \frac{p_2}{w}$$

$$\frac{p_1}{w} - \frac{p_2}{w} = h = x\left(\frac{S_m}{S} - 1\right)$$

Find the expression for inclined Venturi

Venturi Meter - Inclined

Bernoulli's Equation between 1 and 2

$$\frac{p_1}{w} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{w} + \frac{V_2^2}{2g} + z_2$$

$$\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = \frac{V_2^2}{2g} - \frac{V_1^2}{2g}$$

$$h = \frac{V_2^2}{2g} - \frac{V_1^2}{2g} \Rightarrow \mathbf{Q} = C_d \frac{a_1 a_2 \sqrt{2gh}}{\sqrt{a_1^2 - a_2^2}}$$

Manometric equation

$$\frac{p_1}{w} + (z_1 - z_2) + y + x = \frac{p_2}{w} + x\left(\frac{S_m}{S}\right)$$

$$\left(\frac{p_1}{w} + z_1\right) - \left(\frac{p_2}{w} + z_2\right) = h = x\left(\frac{S_m}{S} - 1\right)$$

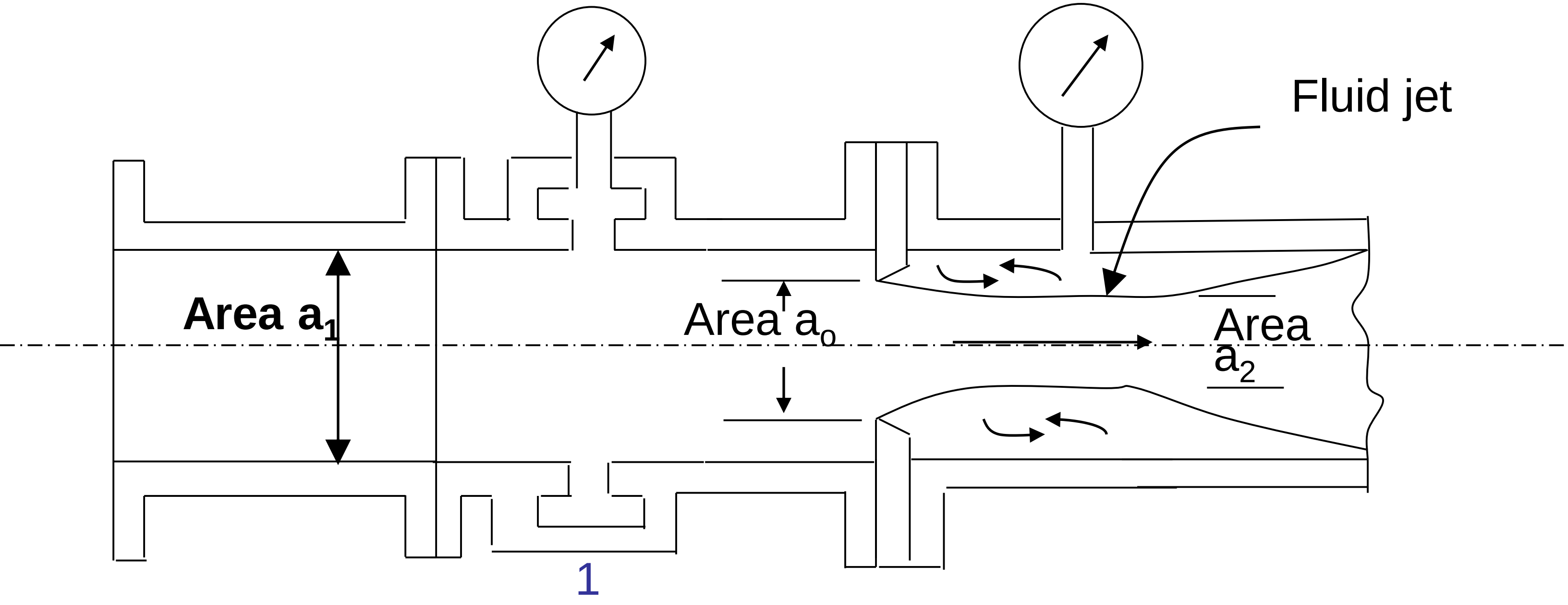
Thus with any position the h may be determined by noting x

The vertical position with flow upward is preferred, why

Venturi Meter

Orifice Meter

- Another device for discharge measurement , works on same principle
- Cheaper arrangement for discharge measurement



- ❑It consists of a circular flat plate with a hole concentric with pipe
- ❑Thickness is 0.05d, edge flat for 0.02d and beveled for 0.03d

Orifice Meter

Apply Bernoulli's

$$\frac{p_1}{w}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{w}+\frac{V_2^2}{2g}+z_2 \Rightarrow \frac{p_1}{w}-\frac{p_2}{w}=\frac{V_2^2}{2g}-\frac{V_1^2}{2g}$$

Thus,

$$h=\frac{V_2^2}{2g}-\frac{V_1^2}{2g} \Rightarrow V_2=(2gh+V_1^2)^{1/2} \Rightarrow$$ **Theoretical velocity**

If the losses are considered $V_2=C_v(2gh+V_1^2)^{1/2}$ **C_v is the coefficient of velocity**

Continuity equation gives

$$Q=a_1V_1=a_2V_2$$

The area of jet a_2 and area of orifice a_o may be given as $\Rightarrow a_2=C_ca_o$

Thus,

$$V_1=V_2C_c\frac{a_o}{a1} \Rightarrow V_2=C_v\left(2gh+V_2^2C_c^2\frac{a_o^2}{a1}\right)^{1/2}$$

Orifice Meter

Solving for V_2 gives

$$V_2 = C_v \left\{ \frac{2gh}{1 - C_v^2 C_c^2 (a_o^2 / a_1^2)} \right\}^{1/2}$$

$$Q = a_2 V_2 = C_c a_o V_2 \ \textbf{and} \ C_c C_v = C_d$$

$$Q = \frac{C_d a_o (2gh)^{1/2}}{\left\{1 - C_d^2 (a_o^2 / a_1^2)\right\}^{1/2}}$$

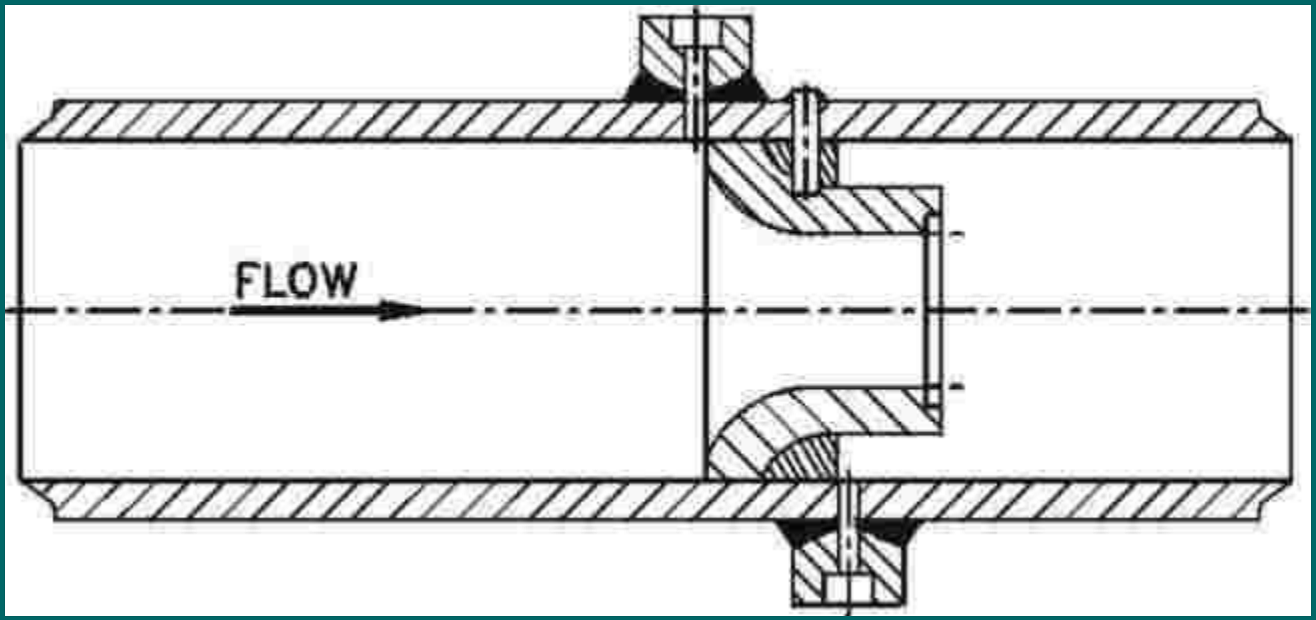
Where C_d is coefficient of discharge

Its usual practice to use a simple expression such that

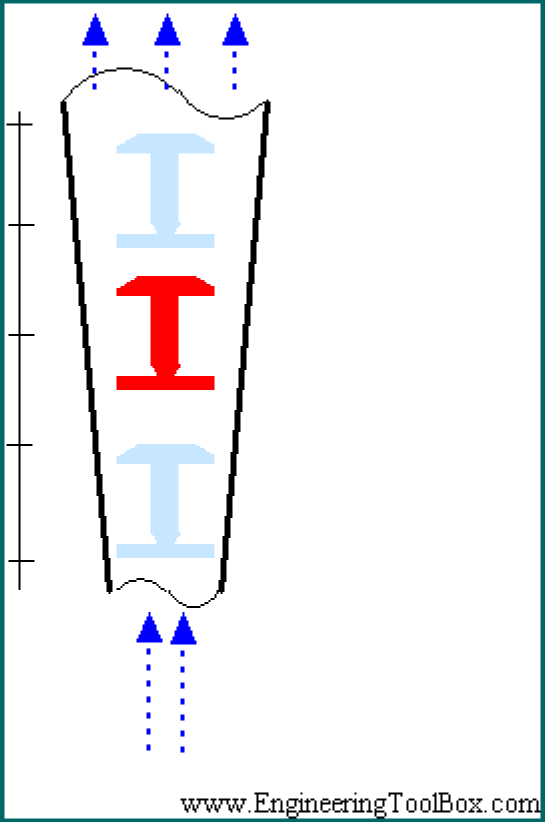
$$C = C_d \frac{\left\{1 - (a_o^2 / a_1^2)\right\}^{1/2}}{\left\{1 - C_d^2 (a_o^2 / a_1^2)\right\}^{1/2}} \Rightarrow Q = \frac{C a_o (2gh)^{1/2}}{\left\{1 - (a_o^2 / a_1^2)\right\}^{1/2}} = \frac{C a_o a_1 (2gh)^{1/2}}{\left\{a_1^2 - a_o^2\right\}^{1/2}}$$

This equation has the same form as Venturi meter

Nozzle Meter

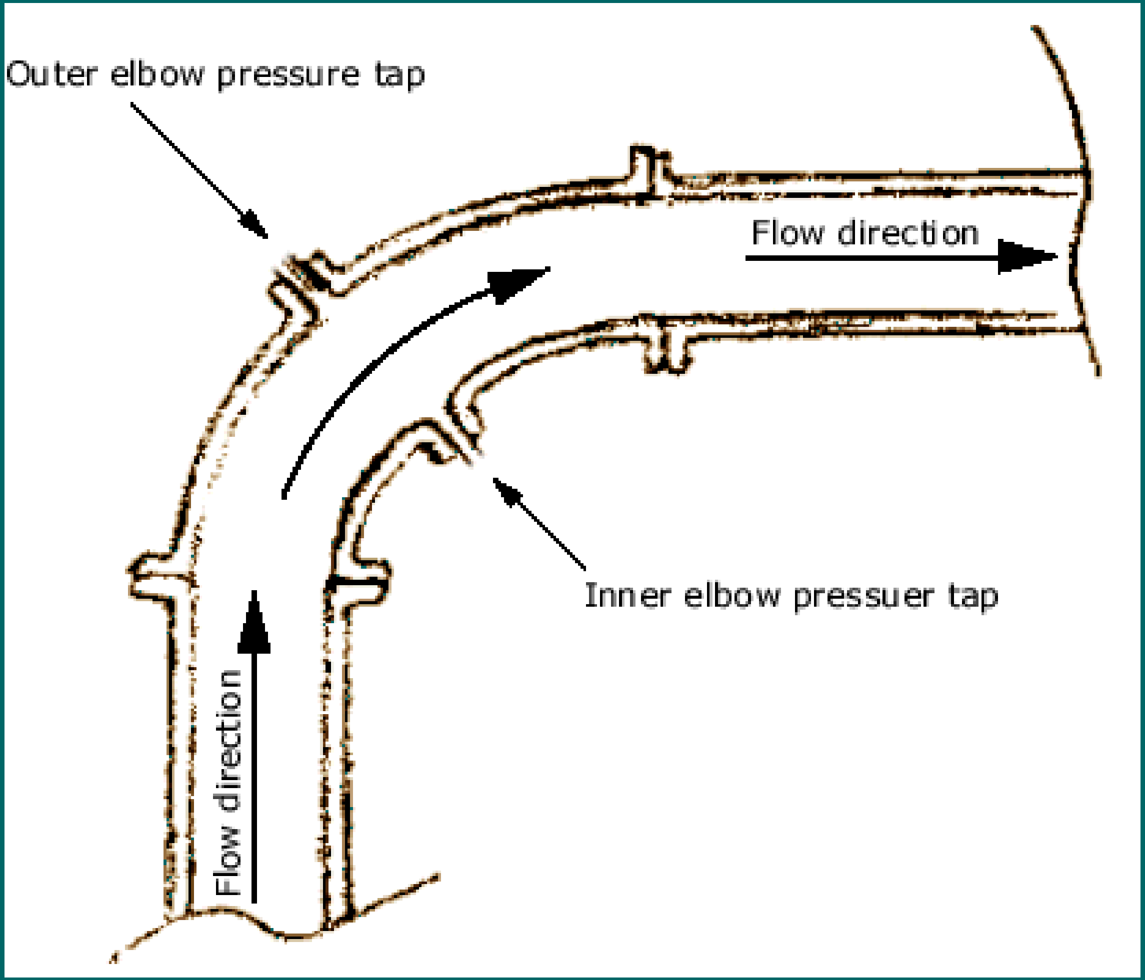


Rota Meter



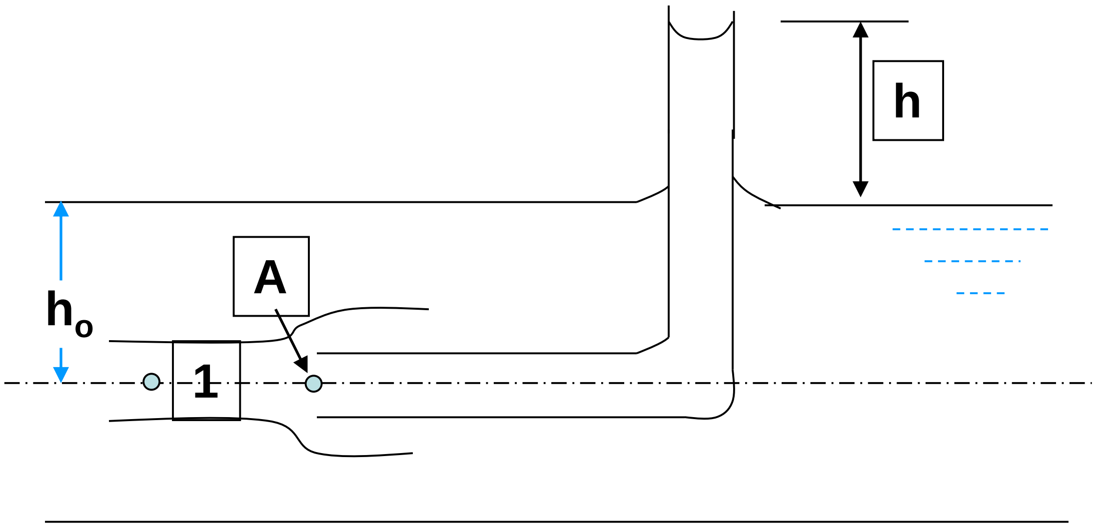
Sample photographs

Elbow Meter



$$Q = C_d A\sqrt{2gh}$$

Pitot Tube



- Simple device for velocity measurement
- Principle – if velocity of flow at a point is reduced to zero (stagnation point), the pressure there is increased due to conversion of KE into pressure energy
- By measuring this pressure rise the velocity at a point can be determined

Bernoulli's Equation between points 1 and A gives

$$h_o + \frac{V^2}{2g} = h_o + h$$

h_o is the static head and h is the dynamic head

$$V = \sqrt{2gh}$$

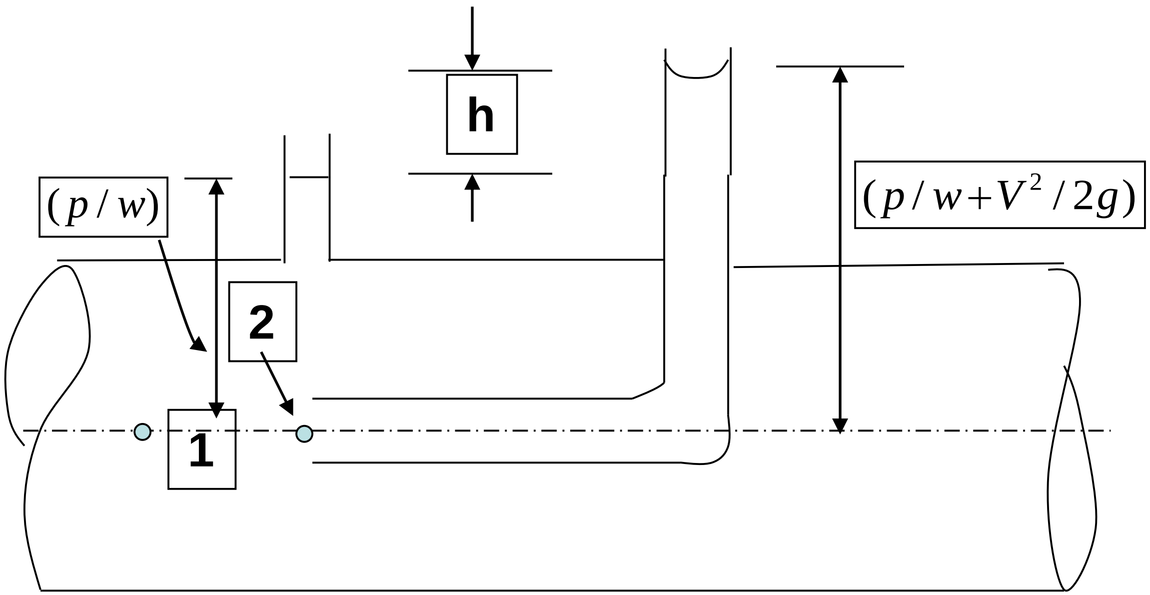
Dynamic pressure head is proportional to a square of velocity

Pitot Tube

- If the losses are considered the velocity is given as

$$V = C\sqrt{2gh}$$

Where C is the coefficient of pitot tube and is around 0.98



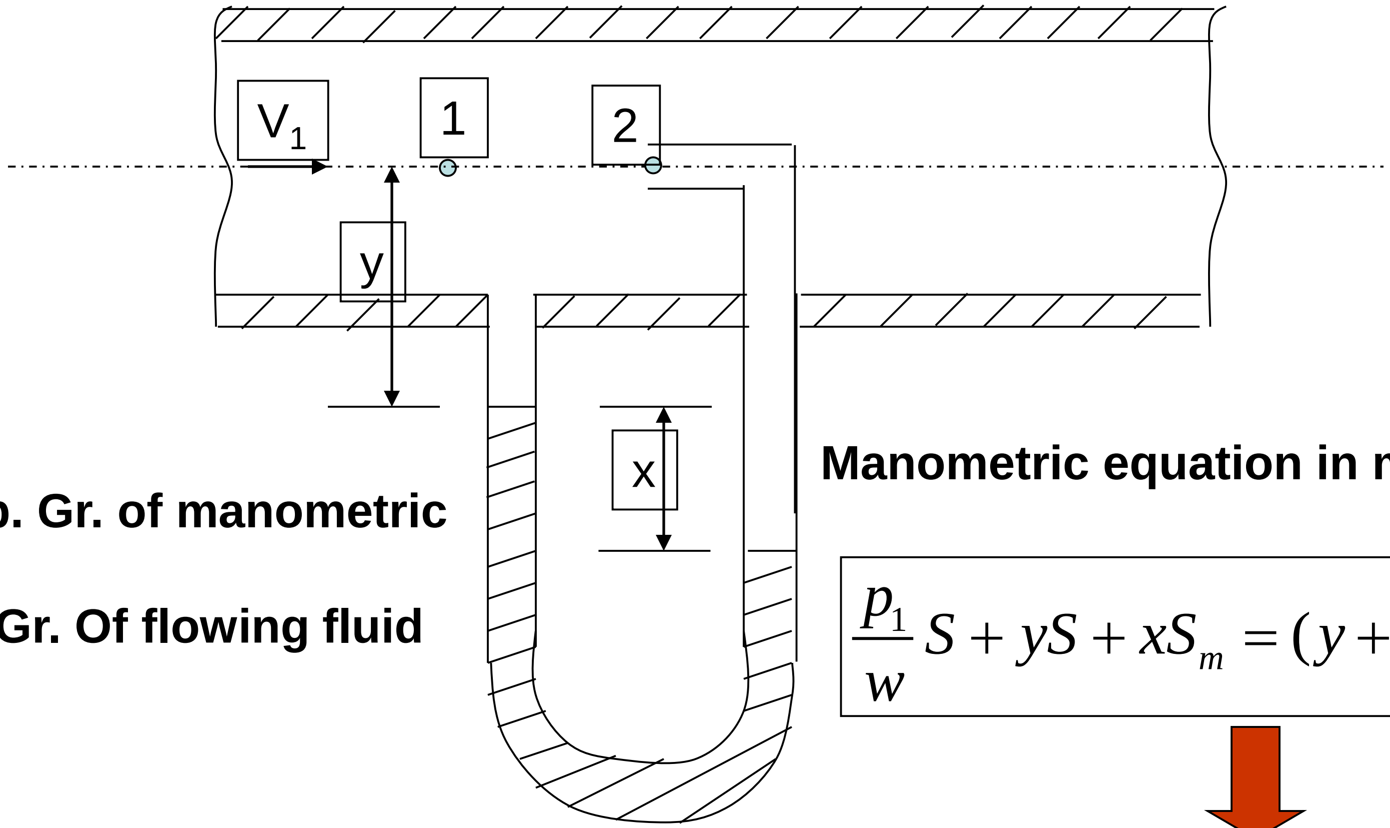
$$\frac{p_1}{w} + \frac{V^2}{2g} = \frac{p_2}{w}$$

$$h = \frac{p_1}{w} - \frac{p_2}{w}$$

$$h = \frac{V^2}{2g}$$

Thus, $V = \sqrt{2gh}$ $\quad V_{act} = C\sqrt{2gh}$

Pitot Tube



- **Sm = sp. Gr. of manometric fluid**
- **S = sp. Gr. Of flowing fluid**

Manometric equation in mts of water

$$\frac{p_1}{w}S + yS + xS_m = (y+x)S + \frac{p_2}{w}S$$

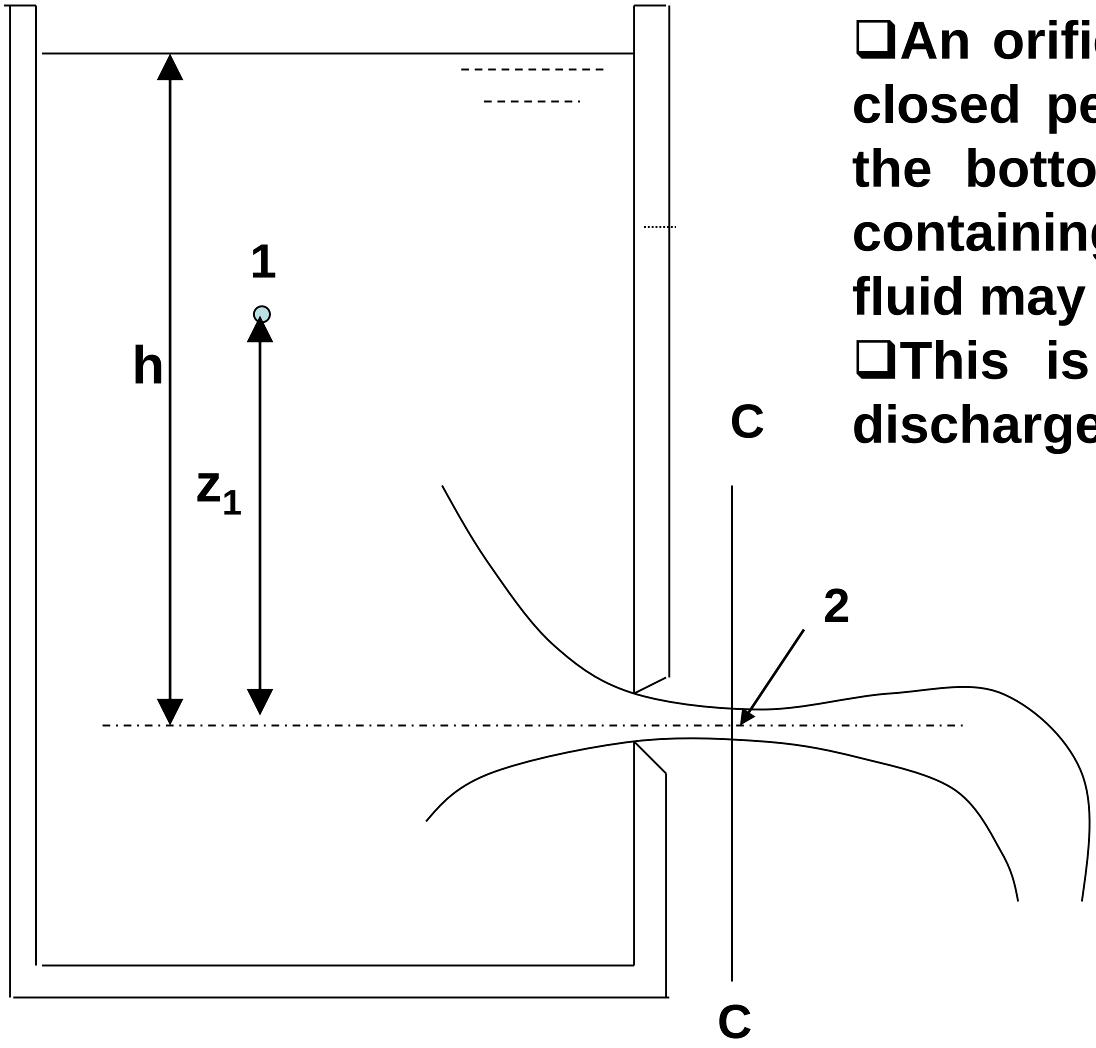
$$\frac{p_1}{w} - \frac{p_2}{w} = h = x\left(\frac{S_m}{S} - 1\right)$$

$$V_{th} = \sqrt{2gx\left(\frac{S_m}{S} - 1\right)} \qquad V_{act} = C\sqrt{2gx\left(\frac{S_m}{S} - 1\right)}$$

Commercial Pitot Tube

Commercial Pitot Tube

Flow through Orifices



❑An orifice is a an opening having a closed perimeter made in the wall at the bottom of the tank or a vessel containing fluid through which the fluid may be discharged

❑This is used for measurement of discharge through tank

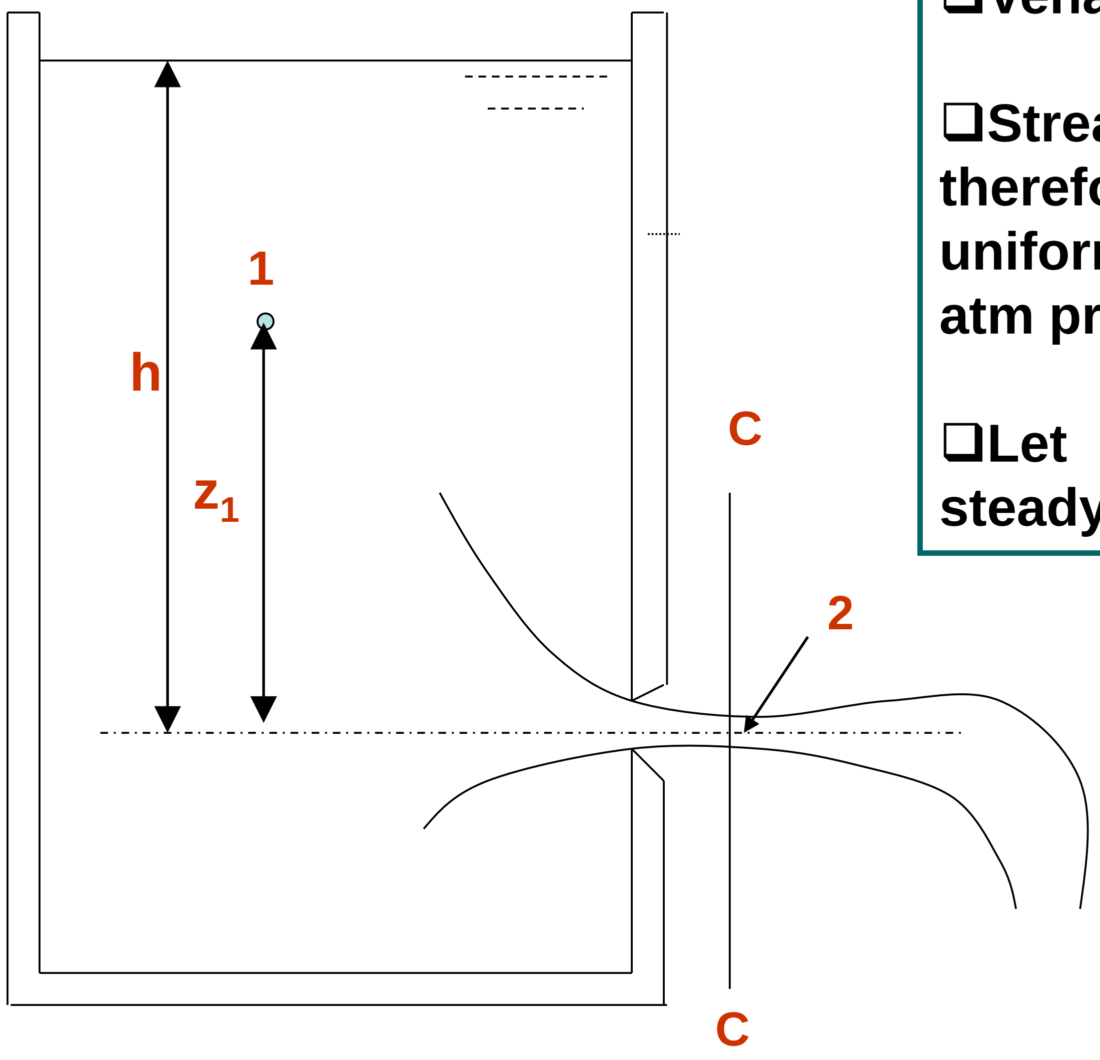
Classification of orifices

- Size – small, large (beyond 5cm diameter)
- Shape – Circular, rectangular, square and triangular
- Upstream edge – sharp edged, Bell mouthed or with round corner
- Discharge conditions – Free discharging

Submerged

Fully submerged **Partially submerged**

Sharp edged Orifice discharging freely



❑Vena contracta formation

❑Streamlines in the jet are parallel therefore the pressure in the jet is uniform throughout and is equal to atm pressure

❑Let the flow through orifice is steady under the constant head h

Consider the sections 1 and 2; 1 being inside the reservoir And 2 being at the center of jet at vena contracta

Sharp edged Orifice discharging freely

Apply BE between 1 and 2

$$\frac{p_1}{w}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{w}+\frac{V_2^2}{2g}+0$$

V1 is the velocity of approach as the fluid approaches the orifice with this velocity

Due to hydrostatic conditions

$$p_1 = p_a + w(h - z_1) \textbf{ and } p_2 = p_a$$

Thus,

$$\frac{p_a + w(h-z_1)}{w}+\frac{V_1^2}{2g}+z_1=\frac{p_a}{w}+\frac{V_2^2}{2g} \Rightarrow \frac{V_2^2}{2g}=\frac{V_1^2}{2g}+h$$

Continuity Equation $Q = a_1 V_1 = a_c V_2$

$$V_2^2 = V_1^2 + 2gh$$

Sharp edged Orifice discharging freely

$$V_2^2 = \frac{a_c^2}{a_1^2} V_2^2 + 2gh \quad \Rightarrow \quad V_2 = \sqrt{\frac{2gh}{1-(a_c/a_1)^2}}$$

V_1 can be assumed to be very small compared to V_2

$$V_2 = \sqrt{2gh}$$

Torricelli's formula

$$V = C_v \sqrt{2gh}$$

Where $C_v = \frac{V}{V_{th}}$ **Varies between 0.95 –0.99**

Coefficient of contraction may be defined as $C_c = \frac{a_c}{a}$

a is area of orifice
a_c is area at VC

Sharp edged Orifice discharging freely

For bell mouthed orifice C_c = 1

$$C_d = \frac{Q_a}{Q_{th}}$$

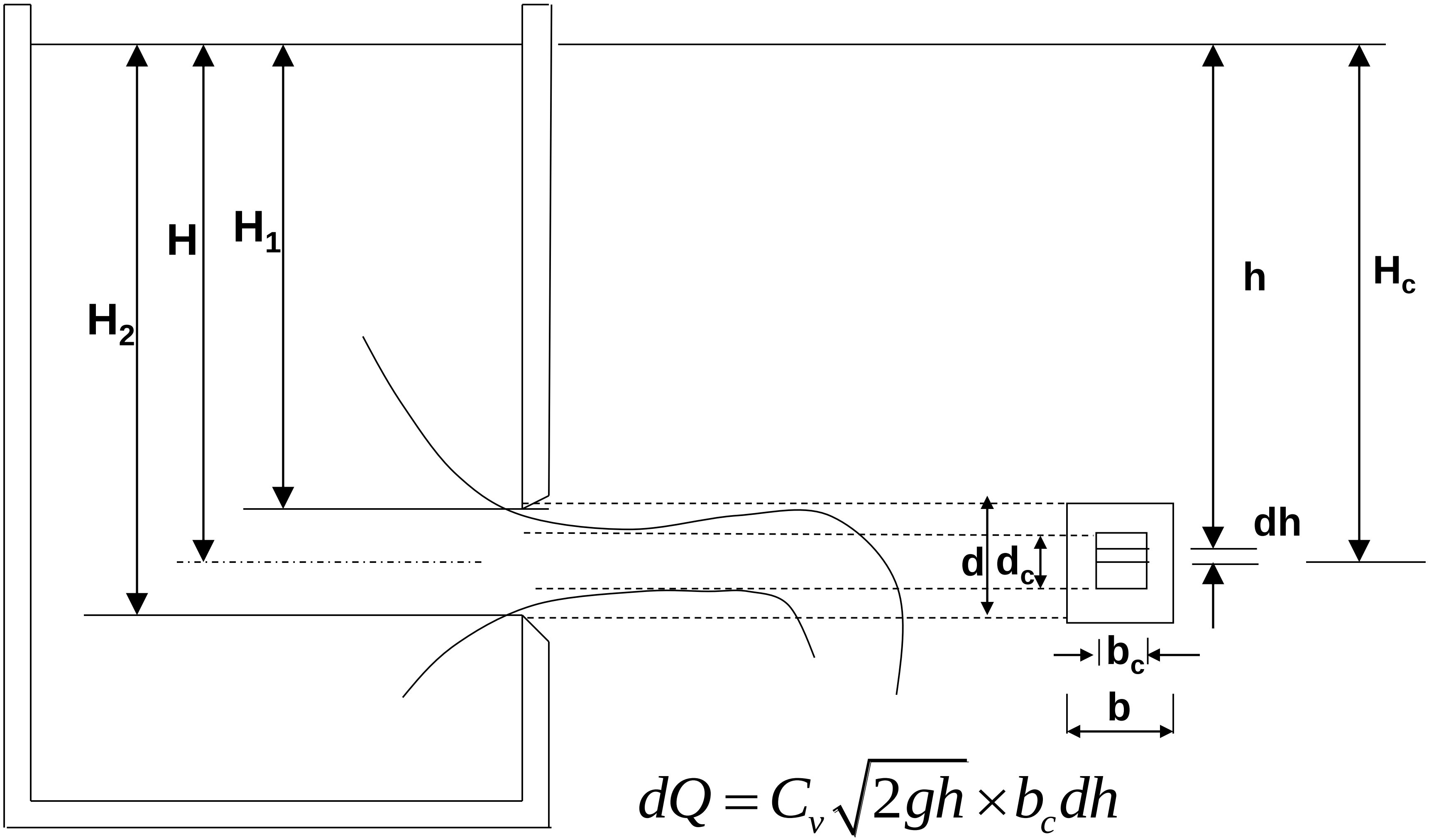
$$Q_a = (a_c V) = C_c a \times C_v \sqrt{2gh}$$

$$Q_{th} = a\sqrt{2gh}$$

$$C_d = \frac{Q_a}{Q_{th}} = \frac{C_c a \times C_v \sqrt{2gh}}{a \times \sqrt{2gh}} \Rightarrow C_d = C_c C_v$$

Thus, $Q_{act} = C_d a \sqrt{2gh}$

Flow through Large Rectangular Orifice



$$dQ = C_v \sqrt{2gh} \times b_c dh$$

Flow through Large Rectangular Orifice

$$Q = C_v b_c \sqrt{2g} \times \int_{(H_c - d_c/2)}^{H_c + d_c/2} h^{1/2} dh$$

$$Q = \frac{2}{3} C_v b_c \sqrt{2g} \times \left\{ \left(H_c + \frac{d_c}{2} \right)^{3/2} - \left(H_c - \frac{d_c}{2} \right)^{3/2} \right\}$$

H_c, b_c or d_c are difficult to determine

$$Q = \frac{2}{3} C_c C_v b \sqrt{2g} \times \left\{ \left(H + \frac{d}{2} \right)^{3/2} - \left(H - \frac{d}{2} \right)^{3/2} \right\}$$

$$Q = \frac{2}{3} C_d b \sqrt{2g} \times \left\{ \left(H + \frac{d}{2} \right)^{3/2} - \left(H - \frac{d}{2} \right)^{3/2} \right\}$$

In terms of H_1 and H_2

$$Q = \frac{2}{3} C_d b \sqrt{2g} \times \left\{ H_2^{3/2} - H_1^{3/2} \right\}$$

For large orifice

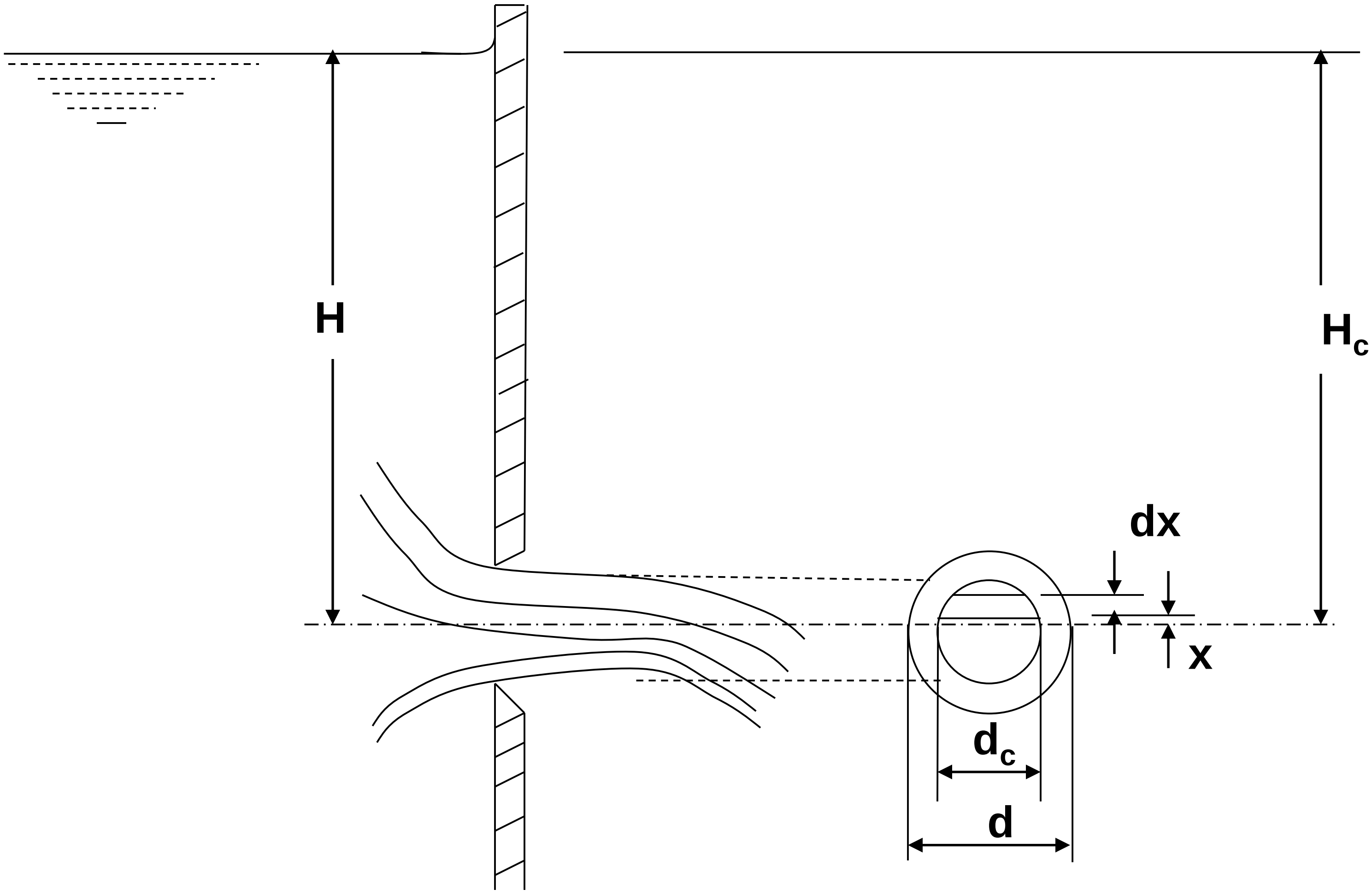
$$Q = C_d b d \sqrt{2g H}$$

Flow through Large Rectangular Orifice

If velocity of approach is considered

$$Q = \frac{2}{3} C_d b \sqrt{2g} \times \left\{ (H_2^{3/2} + V_1^2 / 2) - (H_1^{3/2} + V_1^2 / 2) \right\}$$

Flow through Large Circular Orifice



Area of strip = $\left\{2\sqrt{(d_c/2)^2 - x^2}\right\} \times dx$

Vel. of flow through strip = $C_v\sqrt{2g(H_c - x)}$

Flow through Large Circular Orifice

Discharge through strip $dQ = C_v \sqrt{2g(H_c - x)} \left\{ 2\sqrt{(d_c/2)^2 - x^2} \right\} \times dx$

$$(H_c - x)^{1/2} = H_c^{1/2} - \frac{H_c^{-1/2} x}{2} - \frac{H_c^{-3/2} x^2}{8} - \frac{H_c^{-5/2} x^3}{16} - ....$$ **using binominal theorem**

Thus, $$dQ = 2C_v \sqrt{2g} \left[(H_c - x)^{1/2} = H_c^{1/2} - \frac{H_c^{-1/2} x}{2} - \frac{H_c^{-3/2} x^2}{8} - \frac{H_c^{-5/2} x^3}{16} - .... \right] dx$$

On integrating by varying x between (+d_c/2) to (-d_c/2)

$$Q = C_v \frac{\pi d_c^2}{4} \sqrt{2gH} \left[1 - \frac{d_c^2}{128 H_c^2} - \frac{d_c^4}{1638 H_c^4} - .... \right]$$

$$Q = C_c C_v \frac{\pi d^2}{4} \sqrt{2gH} \left[1 - \frac{d^2}{128 H^2} - \frac{d^4}{1638 H^4} - .... \right]$$

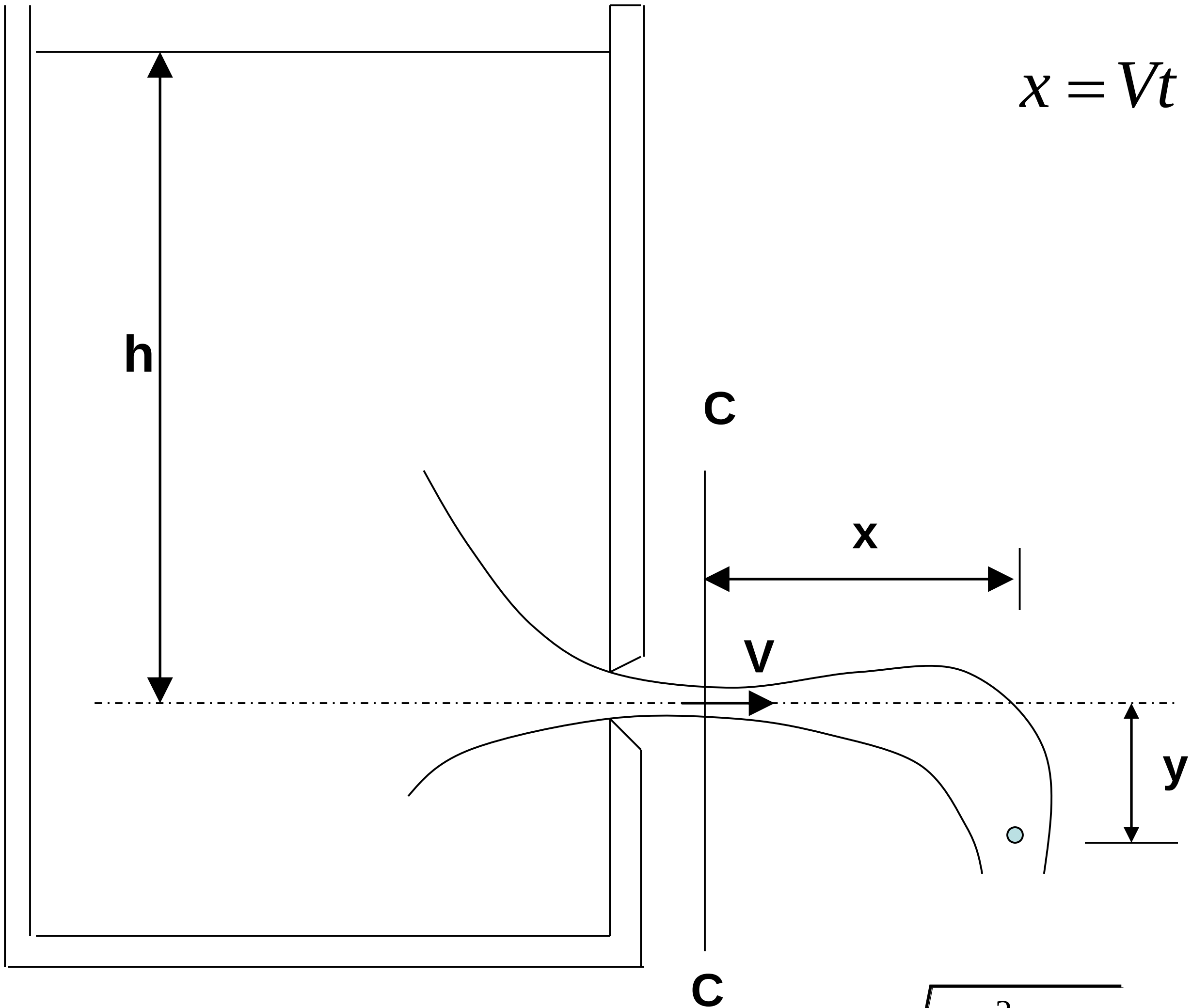
$$Q = C_d \frac{\pi d^2}{4} \sqrt{2gH} \left[1 - \frac{d^2}{128 H^2} - \frac{d^4}{1638 H^4} - .... \right]$$

Flow through Large Circular Orifice

For small orifice the quantity in bracket is less than unity and the actual discharge can be given by

$$Q = C_d \frac{\pi d^2}{4} \sqrt{2gH}$$

Determination of C_v of freely discharging Orifice



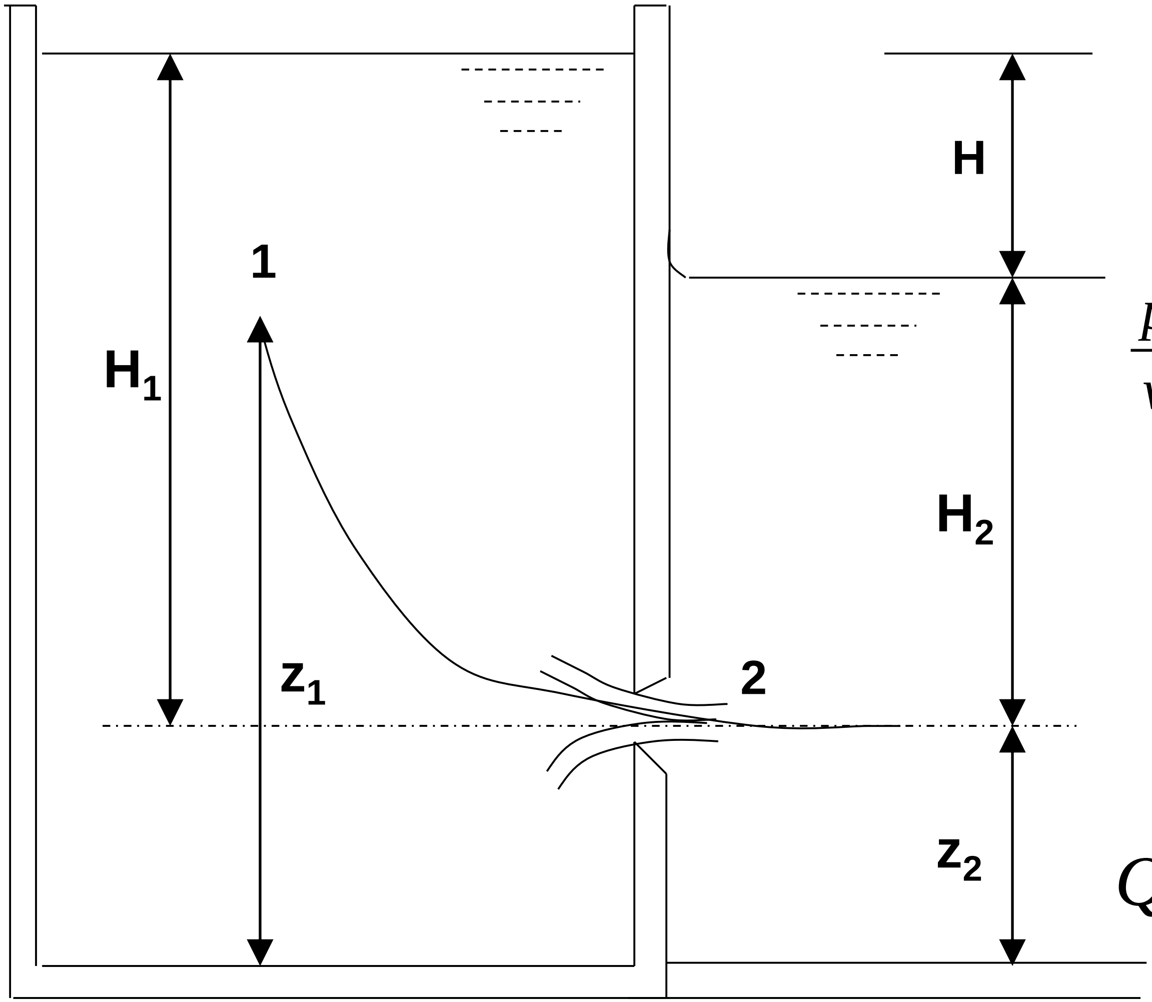
$$x = Vt \textbf{ and } y = \frac{1}{2} g t^2$$

$$y = \frac{1}{2} g \frac{x^2}{V^2}$$

$$V = \sqrt{\frac{g x^2}{2y}}$$

$$C_v = \frac{V}{V_{th}} \Rightarrow C_v = \frac{\sqrt{g x^2 / 2y}}{\sqrt{2gh}} = \sqrt{\frac{x^2}{4hy}}$$

Totally submerged orifice



$$\frac{p_1}{w}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{w}+\frac{V_2^2}{2g}+z_2$$

$$\frac{p_1}{w}=H_1+z_2-z_1 \textbf{ and } \frac{p_2}{w}=H_2$$

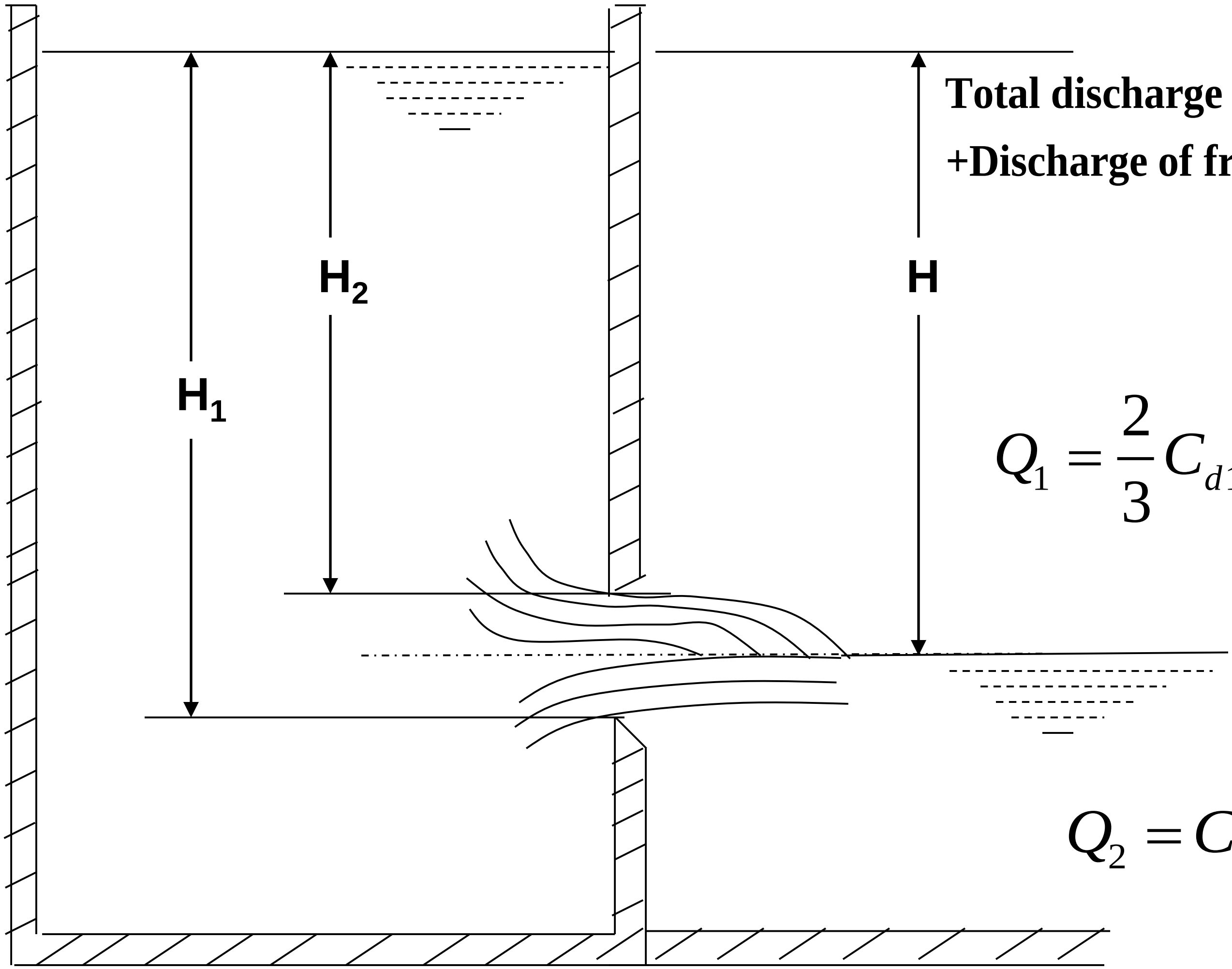
If V_1 is negligible

$$V_2=\sqrt{2g(H_1-H_2)}$$

$$Q=C_d a\sqrt{2g(H_1-H_2)}$$

$$Q=C_d a\sqrt{2gH}$$

Partially submerged orifice



Total discharge = Discharge of submerged part +Discharge of freely discharging orifice

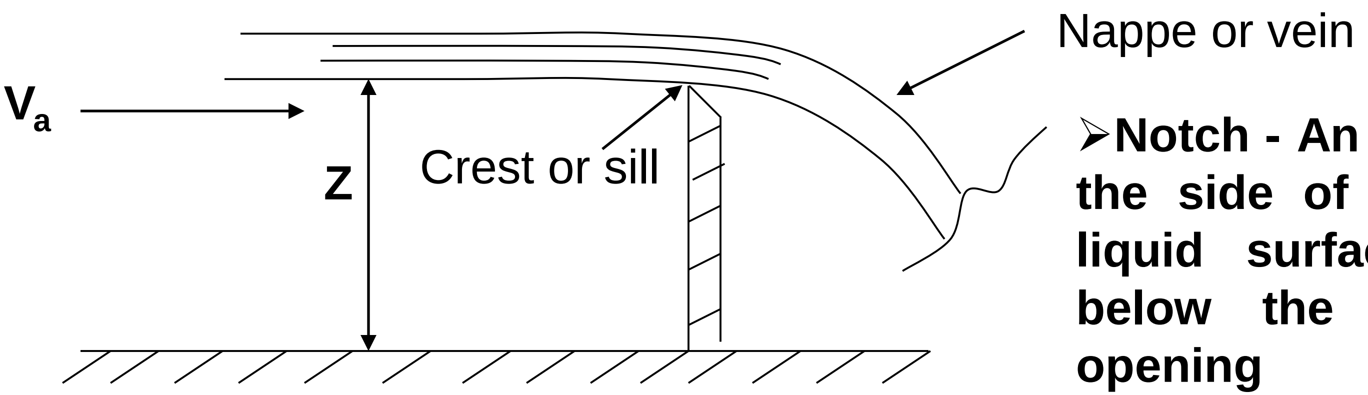
$$Q = Q_1 + Q_2$$

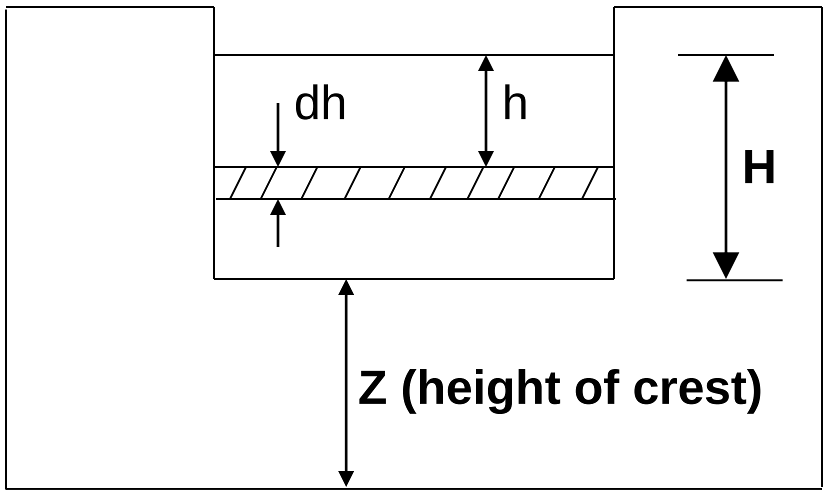
$$Q_1 = \frac{2}{3} C_{d1} b \sqrt{2g} \times \left\{ H^{3/2} - H_2^{3/2} \right\}$$

$$Q_2 = C_{d2} b (H_1 - H) \sqrt{2gH}$$

$$Q = \frac{2}{3} C_{d1} b \sqrt{2g} \times \left\{ H^{3/2} - H_2^{3/2} \right\} + C_{d2} b (H_1 - H) \sqrt{2gH}$$

Notches and Wiers

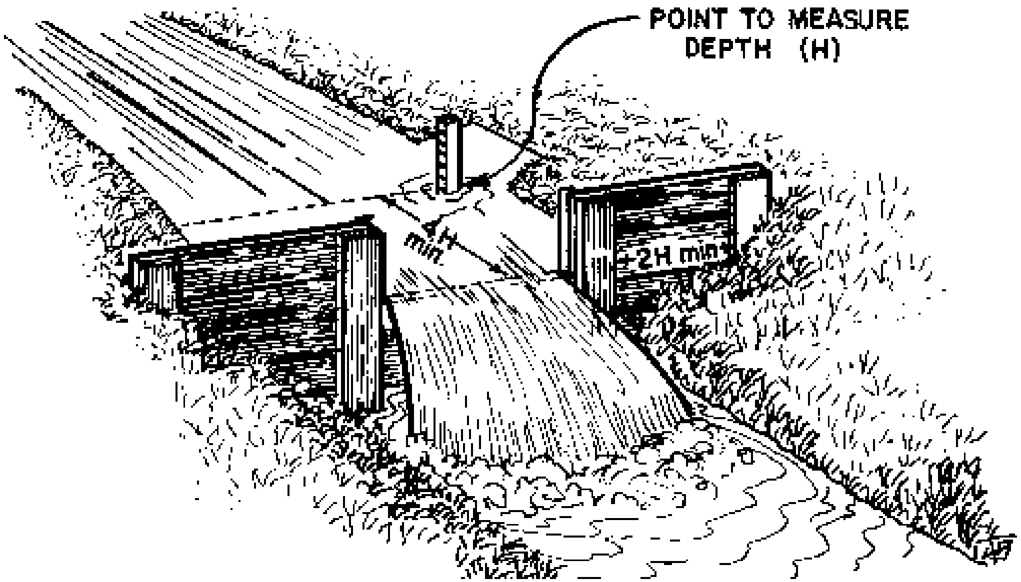




- **Notch - An opening provided in the side of the tank such that liquid surface in the tank is below the top edge of the opening**
- **Notches made up of metallic plates are provided in narrow channels to measure the rate of flow**

- **Weir – Concrete or masonry structure built across the river to raise the level of water on the upstream side and allow the excess water to flow over the entire length to the downstream side**
- **Similar to small dam**

Notches and Wiers



V Notch Weir

Classification of Notches and Wiers

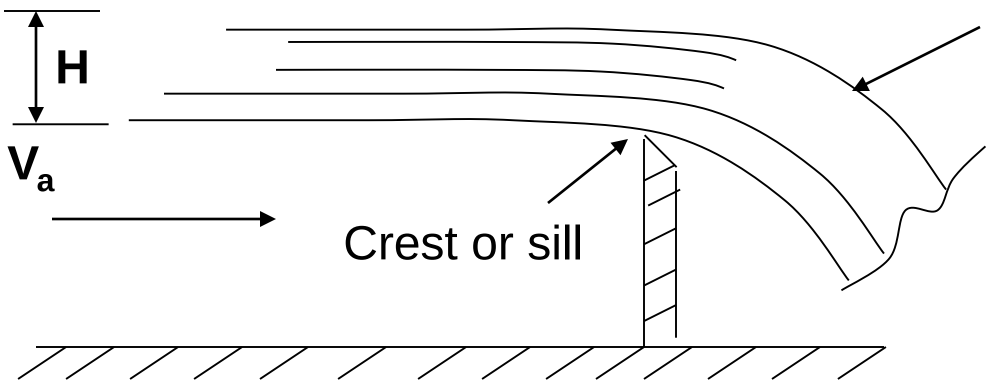
Notches

- According to shape – Rectangular, triangular, trapezoidal, parabolic and stepped
- According to effect of sides on nappe –
  1. Notch with end contraction 2. Notch without end contraction or suppressed notch

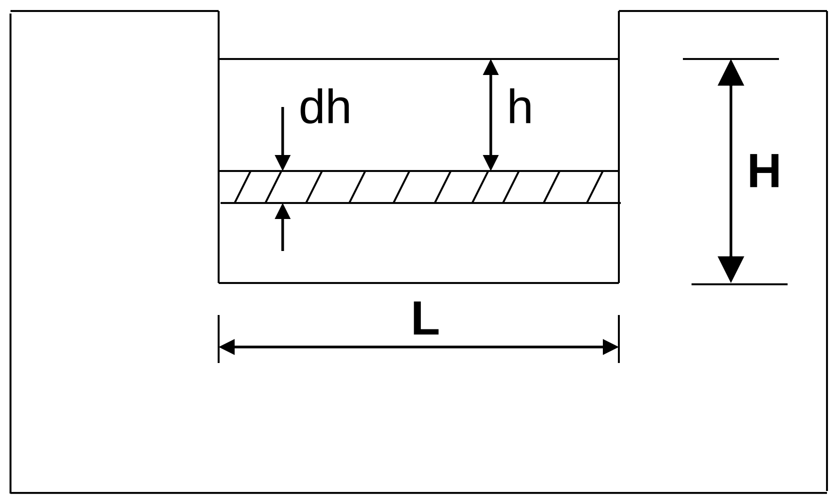
Wiers

- According to shape – Rectangular, triangular, trapezoidal
- According to shape of crest – Sharp crested weir, narrow crested weir broad crested, ogee shaped
- According to effect of sides on nappe –
  1. Notch with end contraction 2. Notch without end contraction or suppressed notch

Flow over Rectangular Sharp Crested Weir or Notch



Consider the elemental strip of thickness dh at a distance of h from free surface



$$\textbf{Area of strip} = L \times dh$$

$$\textbf{Velocity of fluid} = \sqrt{2gh}$$

$$dQ = C_d \times L \times dh \times \sqrt{2gh}$$

$$Q = \int_0^H C_d \times L \times dh \times \sqrt{2gh}$$

$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g} \times H^{3/2}$$

Flow over Rectangular Sharp Crested Weir or Notch

If velocity of approach = V_a , the corresponding head is given as

$$h_a = \frac{V_a^2}{2g}$$

The limits of integration will be h_a to $H+h_a$

Thus

$$Q = \int_{h_a}^{H+h_a} C_d \times L \times \sqrt{2gh} \times dh$$

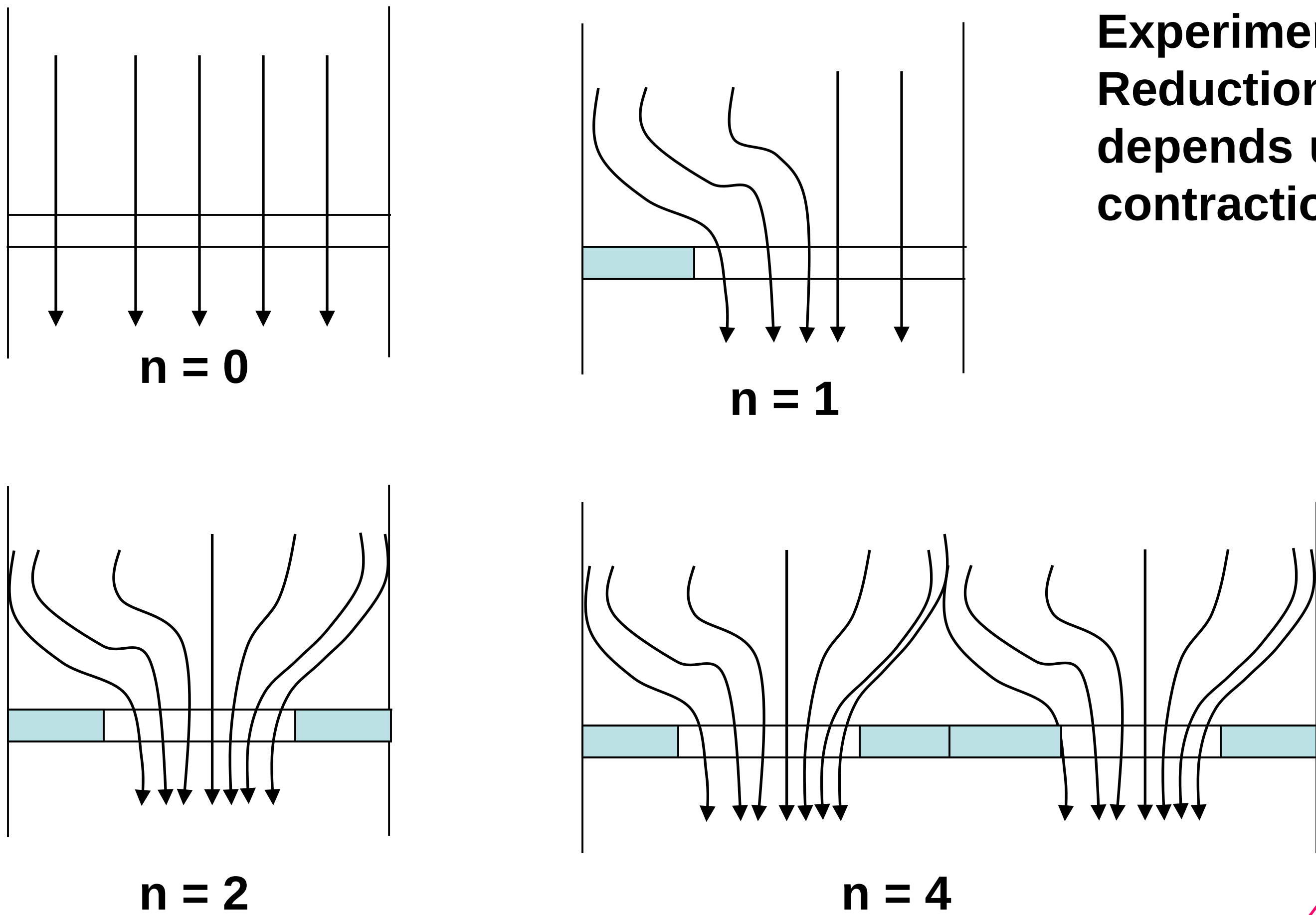
$$Q = \frac{2}{3} C_d \times L \times \sqrt{2g} \times \left[(H+h_a)^{3/2} - h_a^{3/2} \right]$$

$H+h_a$ is known as still water head

The above equation is applicable to suppressed weir or notch for which crest length is equal to the width of channel

Notch with end contraction

End contraction - reduces effective crest length – contraction of nappe – less discharge



Experimental observation – Reduction in crest length depends upon H and for each contraction it is 0.1H

Thus, $$\textbf{Effective crest length of crest} = L - 0.1H$$

C. M. Sewatkar, Faculty, Mech. Engg. Dept., GCOEARA

Notch with end contraction

$$Q = \frac{2}{3} C_d \times \sqrt{2g} \times (L - 0.1H) \times H^{3/2} \quad \textbf{- without } V_a$$

$$Q = \frac{2}{3} C_d \times \sqrt{2g} \times (L - 0.1H_1) \times \left[(H_1)^{3/2} - h_a^{3/2} \right] \quad \textbf{- with } V_a$$

Where,

$$H_1 = (H + h_a) = (H + V_a^2 / 2g)$$

- **Mean value of velocity of approach, Va is given as**

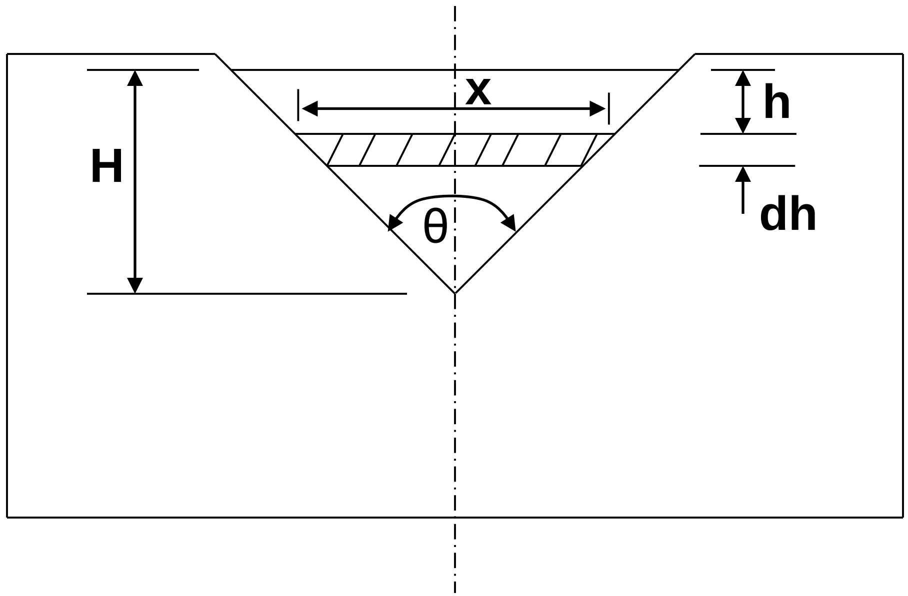
$$V_a = \frac{Q}{B(H+Z)}$$

for the case with end contraction

$$V_a = \frac{Q}{L(H+Z)}$$

for the case of suppressed weir or notch

Triangular Notch



- Triangular notch is preferred over rectangular for low discharges
- The crest length in triangular notch is zero which gives fairly large head for the same discharge as compared to rectangular notch.

$$\frac{x}{2(H-h)} = \tan\frac{\theta}{2} \;\Rightarrow\; x = 2(H-h)\tan\frac{\theta}{2}$$

Area of strip $= xdh = 2(H-h)\tan\frac{\theta}{2}dh$

Velocity $= \sqrt{2gH}$

Therefore

$$dQ = 2(H-h)\tan\frac{\theta}{2}dh\,\sqrt{2gh} \;\Rightarrow\; Q = C_d\int_0^H 2(H-h)\tan\frac{\theta}{2}\,\sqrt{2gH}\,dh$$

Triangular Notch

Hence

$$Q = 2C_d\sqrt{2g}\tan\frac{\theta}{2}\int_0^H (H-h)h^{1/2}dh$$

$$Q = 2C_d\sqrt{2g}\tan\frac{\theta}{2}\left[\frac{2}{3}Hh^{3/2} - \frac{2}{5}h^{5/2}\right]_0^H$$

$$Q = \frac{8}{15}C_d\sqrt{2g}\tan\frac{\theta}{2}H^{5/2}$$

For θ = 90° $Q = \frac{8}{15}C_d\sqrt{2g}H^{5/2}$ **If C_d is assumed to be 0.6** $Q = 1.418H^{5/2}$

If velocity of approach is considered

$$Q = \frac{8}{15}C_d\sqrt{2g}\tan\frac{\theta}{2}\left[(H+h_a)^{5/2} - h_a^{5/2}\right]$$

where, $h_a = \frac{V_a^2}{2g}$

Navier-Stokes Equation

X-momentum equation for laminar flow of an incompressible flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{dp}{dx} + \mu/\rho\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Unsteady term **Convective terms** **Pressure term** **viscous term**

Y- momentum

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{dp}{dy} + \mu/\rho\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right]$$

Z - momentum

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{dp}{dz} + \frac{\mu}{\rho}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right]$$

Vector form or coordinate free form of N-S equation

$$\frac{\partial V}{\partial t} + V.\nabla V = -\frac{1}{\rho}\nabla p + \mu/\rho\,\nabla^2 V$$

Complete N-S Equation

A note on Navier-Stokes Equation

- The N-S equation is used for analyzing the laminar flow
- There is a viscous term in N-S equation apart from the terms in Euler equation
- The viscous term represents the shear forces acting on the fluid particle
- Analytical solution of this equation for 2D and 3D is not possible
- Obtaining the numerical solution of this equation in combination with continuity equation is an important aspect of computational fluid dynamics

Static, Dynamic and Stagnation Pressure

$$\frac{p}{w} + \frac{V^2}{2g} + z = \textbf{Constant} \quad \Rightarrow \quad \frac{p}{\rho} + \frac{V^2}{2} + gz = \textbf{Constant}$$

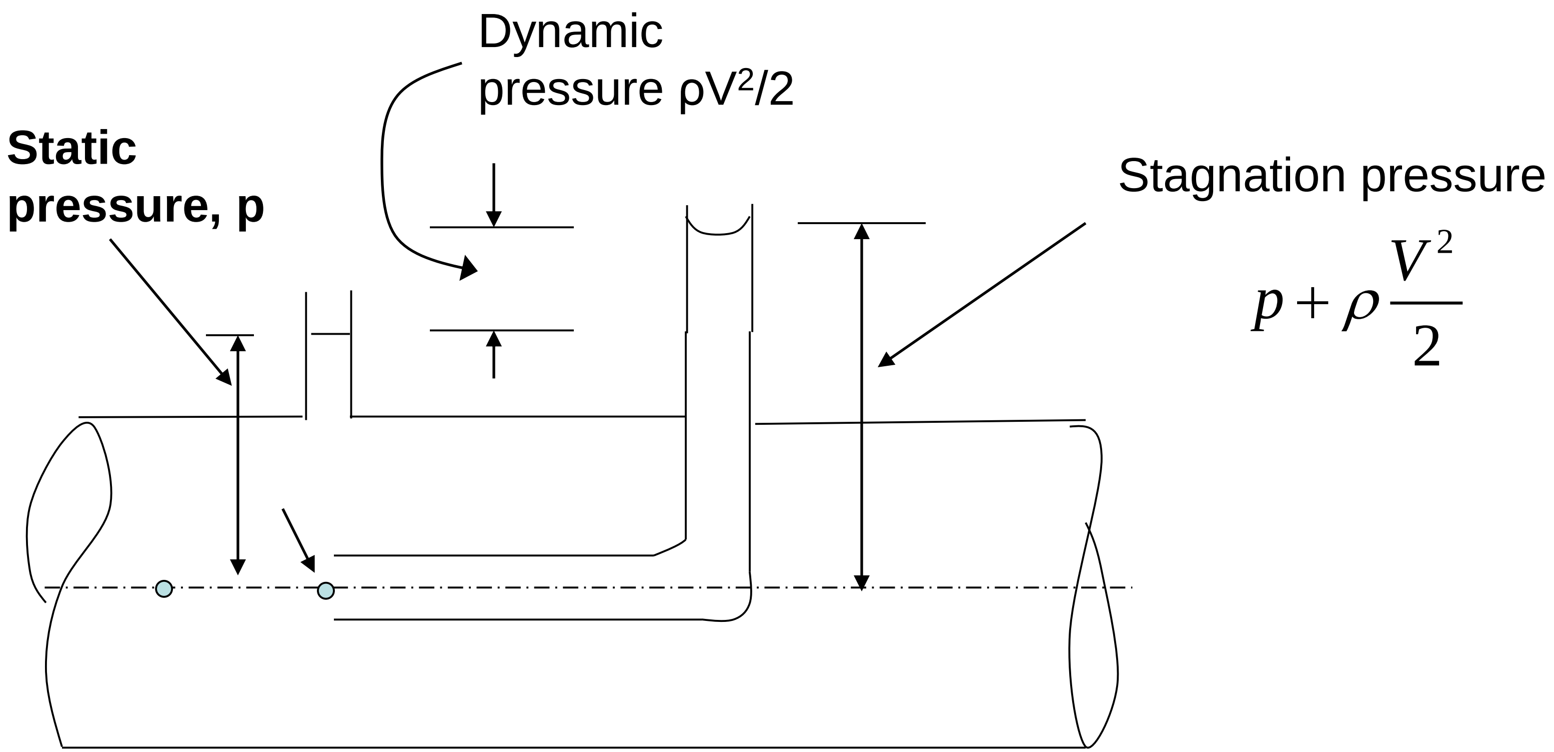
$$p + \rho\frac{V^2}{2} + \rho gz = \textbf{Constant}$$

Each term in the above equation has pressure units and thus each term represents some kind of pressure

- **p is the static pressure; it represents the actual thermodynamic pressure**
- $\rho V^2/2$ **is the dynamic pressure; it represents the pressure rise when the fluid in motion is brought to rest isentropically**
- **ρgz is the hydrostatic pressure; it accounts for elevation effects**

$p + \rho\frac{V^2}{2}$ **represents the stagnation pressure**

Diagrammatic Representation of static, dynamic and stagnation pressure



PROBLEM 1

Calculate the kinetic energy correction factor α for the following velocity distributions in a circular pipe of radius r_0.

1. $$\frac{u}{u_m} = \left(1 - \frac{r}{r_o}\right)$$

2. $$\frac{u}{u_m} = \left[1 - (r / r_o)^2\right]$$

PROBLEM 2

215 liters of gasoline (specific gravity 0.82) flow per second upward in an inclined Venturi meter fitted to a 300 mm diameter pipe. The Venturi meter is inclined at 60° to the vertical and its 150 mm diameter throat is 1.2 m from the entrance along its length. Pressure gages inserted at the entrance and throat show pressures of 0.141 N/mm^2 and 0.077 N/mm^2 respectively. Calculate the discharge coefficient of the Venturi meter.

If instead of pressure gages the entrance and throat are connected to the two limb U-tube mercury manometer, determine its reading in mm of differential mercury column.

PROBLEM 3

The pressure leads from a Pitot-tube mounted on an air craft are connected to a pressure gage in the cockpit. The dial of the pressure gage is calibrated to read the speed in m/s.

The calibration is done on the ground by applying a known pressure across the gage and calculating the equivalent velocity using incompressible Bernoulli's equation and assuming that the density is 1.224 kg/m^3

The gage having been calibrated in this way the air craft is flown at 9200 m, where the density is 0.454 kg/m^3 and ambient pressure is 30000 N/m^2. The gage indicates the velocity of 152 m/s. What is the true speed of the air craft?